# Boosting

- In bagging, we created an ensemble for reducing the variance in high-variance-low-bias (strong) base models

- Boosting is another ensemble method used for reducing the bias in high-bias-low-variance (weak) base models

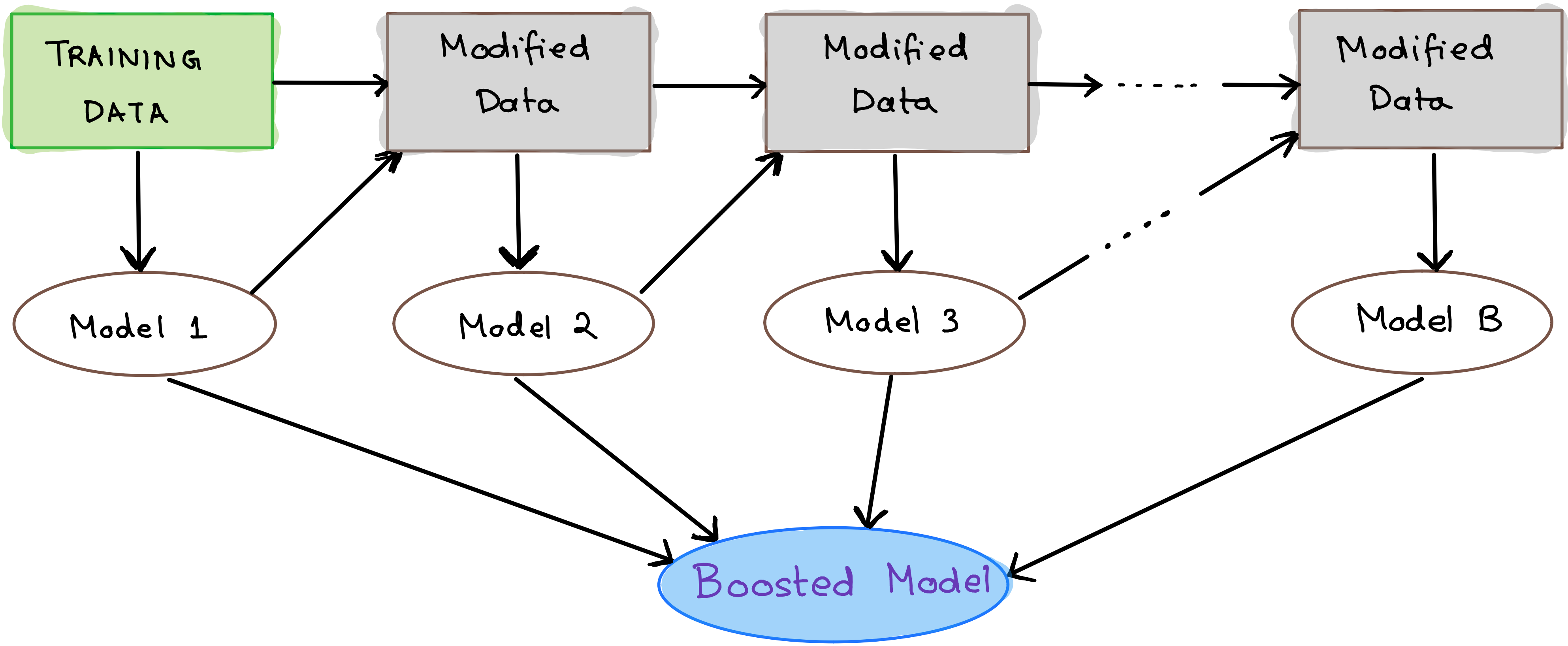
- Intuition:
  - Even a simple (weak) model can typically describe some aspects of the input-output (I/O) relationship
  - Can we then learn an ensemble of "weak models", where each weak model describes some part of the I/O relationship, and combine these models into one "strong model"?

- Boosting shares some similarities with bagging
  - Both use an ensemble of models for combining predictions
  - Both can be used with any regression or classification algorithm

- Difference between bagging and boosting lies in how the base models are being trained
  - In bagging, 'B' identically distributed models are constructed parallely
  - In boosting, the ensemble members are constructed sequentially.

# Sequential Construction in Boosting

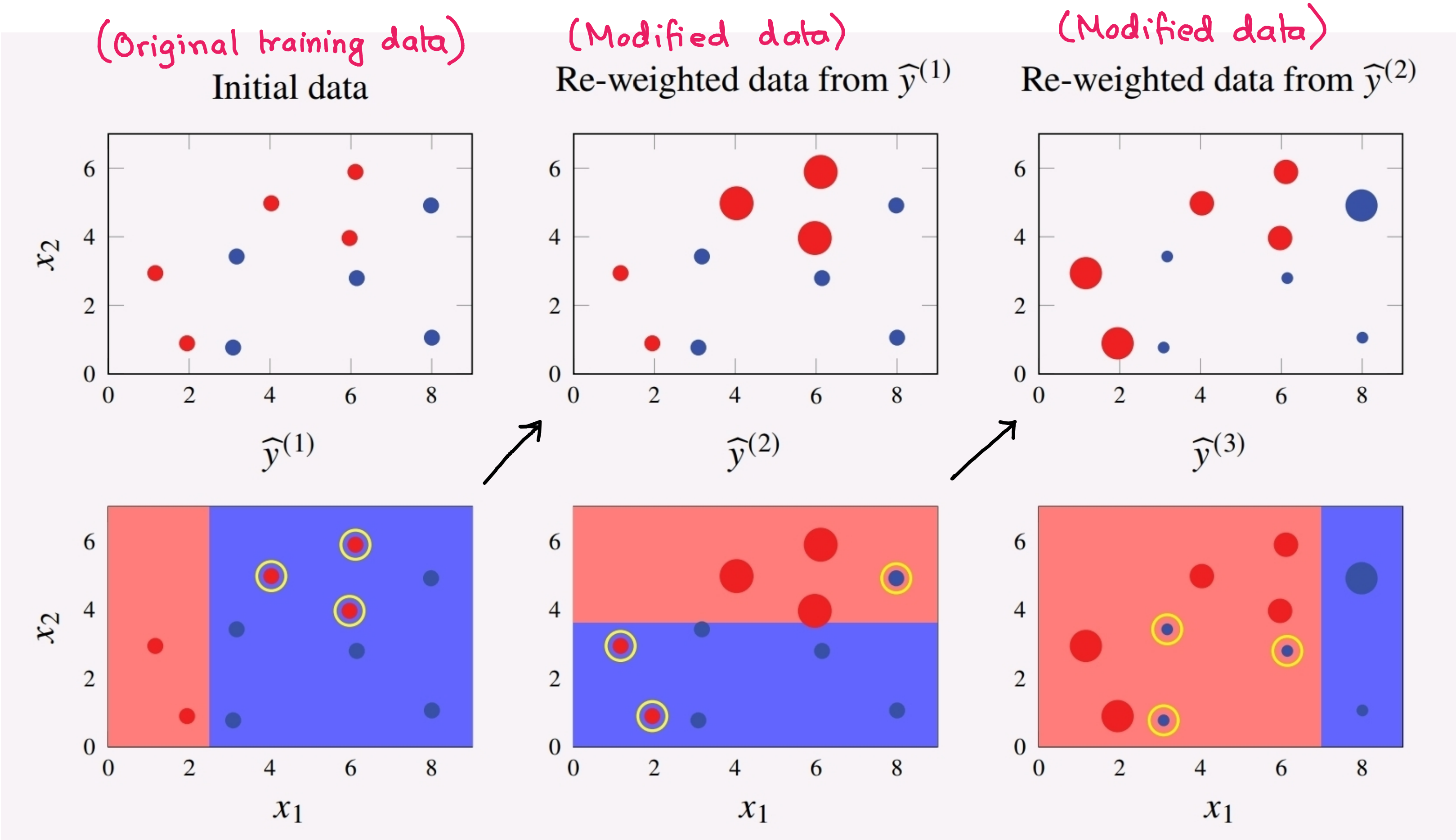
Informally, the sequential construction of ensemble members is done in such a way that each model tries to correct the mistakes made by the previous one



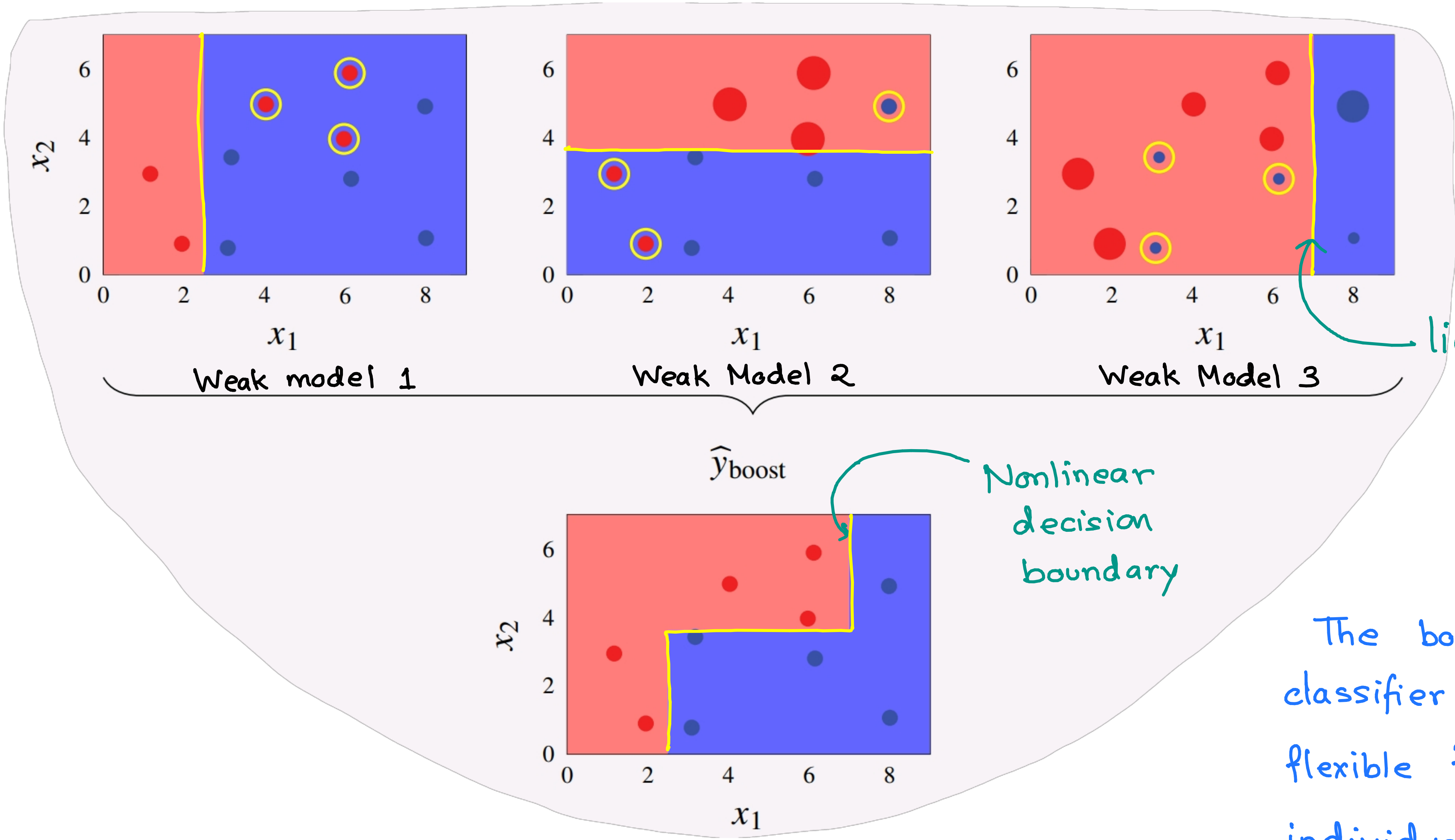
# Example of sequential construction in boosting

Consider a binary classification problem with 2D input $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

- There are $N = 10$ datapoints, 5 from each class
- A classification tree of depth one (weak model) is used as the base classifier (splits into two regions)




Weak model 1     Weak model 2     Weak model 3




Boosted model

The boosted classifier is more flexible than the individual classifiers

The final classifier $\hat{y}_{\text{boost}}(\underline{x})$ = Weighted majority vote of the three weak decision trees

# Boosting Procedure (for classification)

**Input:** Training set $T = \{ \underline{x}^{(i)}, y^{(i)} \}_{i=1}^{N}$

**Output:** Boosted predictions $\hat{y}_{boost}(\underline{x})$

1. Assign weights $w_i^{(1)} = 1/N$ to all data points

2. For $b = 1$ to $B$

   - Train a weak classifier $\hat{y}^{(b)}(\underline{x})$ on the weighted training data $\{(\underline{x}^{(i)}, y^{(i)}, w_i^{(b)})\}_{i=1}^{N}$

   - Update the weights $\{w_i^{(b+1)}\}_{i=1}^{N}$ from $\{w_i^{(b)}\}_{i=1}^{N}$ :

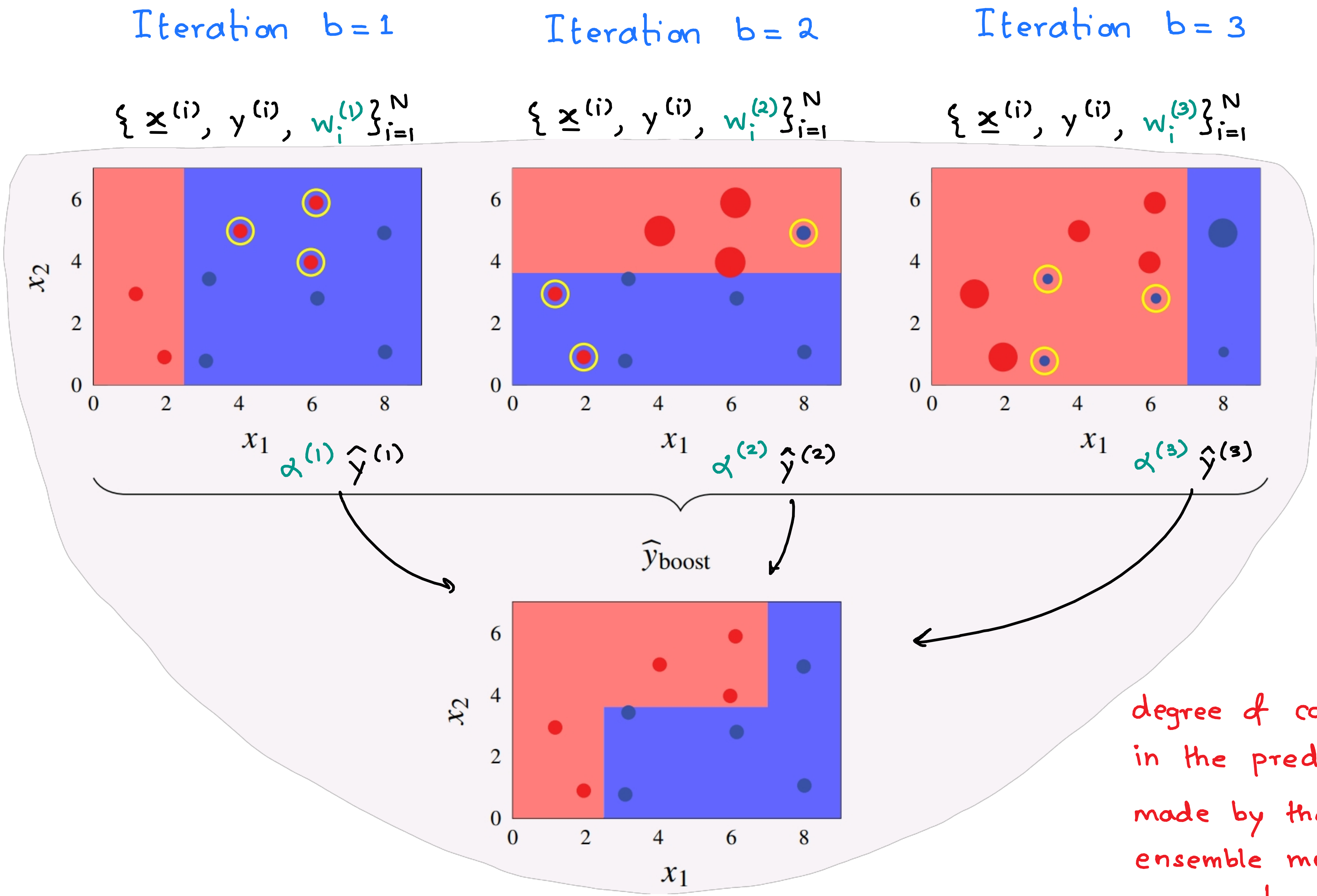
     → Increase weights for all points misclassified by $\hat{y}^{(b)}(\underline{x})$

     → Decrease weights for all points correctly classified by $\hat{y}^{(b)}(\underline{x})$

3. The predictions from the 'B' classifiers, $\hat{y}^{(1)}(\underline{x}), \hat{y}^{(2)}(\underline{x}), \ldots, \hat{y}^{(B)}(\underline{x})$, are combined using a weighted majority vote:

$$\boxed{\hat{y}_{boost}(\underline{x}) = \text{sign}\left( \sum_{b=1}^{B} \alpha^{(b)} \hat{y}^{(b)}(\underline{x}) \right)}$$

$\alpha^{(b)} > 0$ always




degree of confidence in the predictions made by the 'b' th ensemble member

$$\hat{y}_{\text{boost}}(\underline{x}) = \text{sign}\left(\sum_{b=1}^{3} \alpha^{(b)} \hat{y}^{(b)}(\underline{x})\right)$$

- How do we reweight the data, $w_i^{(b)}$s ?
- How are the coefficients $\alpha^{(1)}, \dots, \alpha^{(B)}$ computed ?

## AdaBoost (Adaptive Boosting)

- It is the first successful implementation of the idea of boosting
- We will restrict our focus to binary classification, but boosting is also applicable to multi-class classification & regression problems
- Output of the AdaBoost classifier:

+1/−1 from individual members

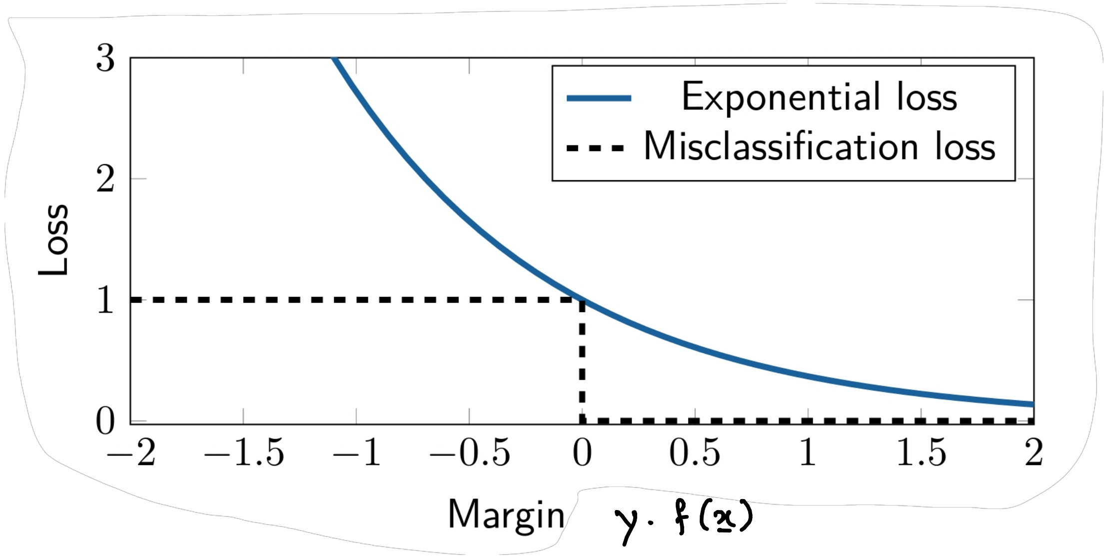
$$\hat{y}_{boost}(\underline{x}) = \text{sign}\left\{ \sum_{b=1}^{B} \alpha^{(b)} \hat{y}^{(b)}(\underline{x}) \right\}$$

- The training of an AdaBoost classifier follows the general form of a binary classifier $y = \text{sign}\{f(\underline{x})\}$
  - The class predictions are obtained by thresholding $f(\underline{x})$ at zero
  - In AdaBoost, they are obtained by thresholding the weighted sum of predictions made by all ensemble members

# Exponential Loss in AdaBoost

- AdaBoost uses exponential loss ← because it results in convenience in calculations

$$L(y, \hat{y}) = \exp\Big(-\underbrace{y \cdot \overbrace{f(\underline{x}; \underline{\theta})}^{\hat{y}}}_{\text{Margin}}\Big)$$




- The ensemble members are added one at a time, and when the 'b' th member is added, it is done to minimize the exponential loss of the entire ensemble constructed so far

# Training of AdaBoost Classifier

- Lets write the boosted classifier after 'b' iterations

$$\hat{y}^{(b)}_{boost}(\underline{x}) = \text{sign}\left\{ \underbrace{\sum_{j=1}^{b} \alpha^{(j)} \hat{y}^{(j)}(\underline{x})}_{f^{(b)}(\underline{x})} \right\}$$

$$= \text{sign}\left\{ f^{(b)}(\underline{x}) \right\}$$

Iter 1    Iter 2

Model 1    Model 2    ....    Model B

$\hat{y}^{(1)}$    $\hat{y}^{(2)}$

$\alpha^{(1)}$    $\alpha^{(2)}$

$$\hat{y}^{(2)}_{boost} = \sum_{j=1}^{2} \alpha^{(j)} \hat{y}^{(j)}$$

output of 'b'th ensemble

- We can express $f^{(b)}(x)$ iteratively : $f^{(b)}(\underline{x}) = f^{(b-1)}(\underline{x}) + \alpha^{(b)} \hat{y}^{(b)}(\underline{x})$

- The ensemble members as well as the coefficients $\alpha^{(b)}$ are constructed sequentially
  - At the 'b' iteration, function $f^{(b-1)}(\underline{x})$ is known and kept fixed
  - Only $\alpha^{(b)}$ and the 'b'th model $\hat{y}^{(b)}(\underline{x})$ is learned
  - This is also called "GREEDY" construction

- $$f^{(b)}(\underline{x}) = f^{(b-1)}(\underline{x}) + \alpha^{(b)} \hat{y}^{(b)}(\underline{x}) \qquad [\, f^{(0)}(\underline{x}) = 0 \,]$$

- Training is done by minimizing the exponential loss of the data:

$$\left(\hat{\alpha}^{(b)}, \hat{y}^{(b)}(\underline{x})\right) = \underset{(\alpha, \hat{y})}{\arg\min} \sum_{i=1}^{N} L\left(y^{(i)}, f^{(b)}(\underline{x}^{(i)})\right)$$

$$= \underset{(\alpha, \hat{y})}{\arg\min} \sum_{i=1}^{N} \exp\left(-y^{(i)} \cdot f^{(b)}(\underline{x}^{(i)})\right)$$

$$= \underset{(\alpha, \hat{y})}{\arg\min} \sum_{i=1}^{N} \exp\left(-y^{(i)} \cdot \left(f^{(b-1)}(\underline{x}^{(i)}) + \underbrace{\alpha \hat{y}(\underline{x}^{(i)})}_{\text{Unknown}}\right)\right)$$

$$= \underset{(\alpha, \hat{y})}{\arg\min} \sum_{i=1}^{N} \underbrace{\exp\left(-y^{(i)} f^{(b-1)}(\underline{x}^{(i)})\right)}_{= w_i^{(b)}} \exp\left(-y^{(i)} \alpha \hat{y}(\underline{x}^{(i)})\right)$$

- Weights for individual data points in training set for 'b' th iteration

$$w_i^{(b)} \stackrel{\text{def}}{=} \exp\left(-y^{(i)} f^{(b-1)}(\underline{x}^{(i)})\right)$$

- Weights $w_i^{(b)} = \exp\left(-y^{(i)} f^{(b-1)}(\underline{x}^{(i)})\right)$

- Note that the weights $\{w_i^{(b)}\}_{i=1}^N$ are independent of $\alpha^{(b)}$ & $\hat{y}^{(b)}(\underline{x})$

  - When learning $\hat{y}^{(b)}(\underline{x})$ and $\alpha^{(b)}$ by solving the loss minimization, we can consider $\{w_i^{(b)}\}_{i=1}^N$ as constants

Constant

$$(\hat{\alpha}^{(b)}, \hat{y}^{(b)}(\underline{x})) = \arg\min_{(\alpha, \hat{y})} \sum_{i=1}^N w_i^{(b)} \exp\left(-y^{(i)} \alpha \hat{y}(\underline{x}^{(i)})\right)$$

- Rewrite the objective function as

Indicator function returns 0/1

$$\sum_{i=1}^N w_i^{(b)} \exp\left(-y^{(i)} \alpha \hat{y}(\underline{x}^{(i)})\right) = e^{-\alpha} \underbrace{\sum_{i=1}^N w_i^{(b)} \mathbb{I}\{y^{(i)} = \hat{y}(\underline{x}^{(i)})\}}_{= W_c}$$

correct

incorrect

$$+ e^{\alpha} \underbrace{\sum_{i=1}^N w_i^{(b)} \mathbb{I}\{y^{(i)} \neq \hat{y}(\underline{x}^{(i)})\}}_{= W_e}$$

* $\hat{y}(\underline{x}^{(i)})$ is the ensemble member we are to learn here

- Rewritting the objective function:

$$\underbrace{\sum_{i=1}^{N} w_i^{(b)} \exp\left(-y^{(i)}\alpha\,\hat{y}(\underline{x}^{(i)})\right)}_{\text{"OBJ"}} = \underbrace{W_c}_{\substack{\text{Correctly} \\ \text{classified} \\ \text{data points}}} + \underbrace{W_e}_{\substack{\text{Incorrectly} \\ \text{classified} \\ \text{data points}}}$$

$$W_c = e^{-\alpha} \sum_{i=1}^{N} w_i^{(b)} \, \mathbb{I}\{y^{(i)} = \hat{y}(\underline{x}^{(i)})\}$$

$$W_e = e^{\alpha} \sum_{i=1}^{N} w_i^{(b)} \, \mathbb{I}\{y^{(i)} \neq \hat{y}(\underline{x}^{(i)})\}$$

- Let $W = W_c + W_e$ be the total sum of weights

$$= \sum_{i=1}^{N} w_i^{(b)}$$

- The "OBJ" is minimized in two stages:
  - first w.r.t. $\hat{y}$
  - then w.r.t $\alpha$

- This is possible because the argument $\hat{y}$ turns out to be independent of the actual value of $\alpha$ $(>0)$

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- To see this, note that we can write the "OBJ" function

$$\text{"OBJ"} = e^{-\alpha} W_c + e^{\alpha} W_e$$

$$= e^{-\alpha}(W - W_e) + e^{\alpha} W_e = e^{-\alpha} W + (e^{\alpha} - e^{-\alpha}) W_e$$

$W = \sum_{i=1}^{N} w_i^{(b)}$ ← independent of $y^{(i)}$

weights of incorrectly classified points

- Minimizing "OBJ" is equivalent to minimizing $W_e$ w.r.t. $\hat{y}$

$$\hat{y}^{(b)} = \arg\min_{\hat{y}} \sum_{i=1}^{N} w_i^{(b)} \mathbb{I}\{y^{(i)} \neq \hat{y}(\underline{x}^{(i)})\}$$ ← misclassification loss

- Minimizing "OBJ" is equivalent to minimizing $W_e$ w.r.t. $\hat{y}$

$$\hat{y}^{(b)} = \arg\min_{\hat{y}} \sum_{i=1}^{N} \underbrace{w_i^{(b)} \mathbb{I}\{y^{(i)} \neq \hat{y}(\underline{x}^{(i)})\}}_{\text{weighted misclassification loss for the ith data point}}$$

- So, the 'b'th ensemble member should be trained by minimizing the weighted misclassification loss for all data points
  - This resembles standard training of classifiers, except for the weights $w_i^{(b)}$, which boils down to weighing the loss for each data point

- The intuition for weights $w_i^{(b)}$ is that, at iteration 'b', we should focus our attention on data points previously misclassified in order to "correct the mistakes" made by the ensemble of the first (b-1) classifiers

- Once the 'b' th ensemble member, $\hat{y}^{(b)}(\underline{x})$, has been trained we then need to learn coefficient $\alpha^{(b)}$

- It is done by minimizing the "OBJ" w.r.t $\alpha$

$$\alpha^{(b)} = \underset{\alpha}{\operatorname{argmin}} \; e^{\alpha} W + (e^{\alpha} - e^{-\alpha}) W_e$$

– Differentiate w.r.t. $\alpha$ and set the derivative to zero

$$\Rightarrow \quad -\alpha e^{-\alpha} W + \alpha (e^{\alpha} + e^{-\alpha}) W_e = 0$$

$$\Leftrightarrow \quad W = (e^{2\alpha} + 1) W_e$$

$$\Leftrightarrow \quad \alpha = \frac{1}{2} \ln \left( \frac{W}{W_e} - 1 \right)$$

- Optimal value of $\alpha$ : $\alpha = \frac{1}{2} \ln \left( \frac{W}{W_e} - 1 \right)$

- By defining $E^{(b)}_{train} = \frac{W_e}{W} = \sum_{i=1}^{N} \frac{w_i^{(b)}}{\sum_{j=1}^{N} w_j^{(b)}} \mathbb{I}\{ y^{(i)} \neq \hat{y}^{(b)}(\underline{x}^{(i)}) \}$

  to be the weighted misclassification error for the 'b' th classifier we can express the optimal value of $\alpha$ as:

$$\alpha^{(b)} = \frac{1}{2} \ln \left( \frac{1 - E^{(b)}_{train}}{E^{(b)}_{train}} \right)$$

- $\alpha^{(b)}$ depends upon the training error of the 'b' th ensemble member
  - Hence, $\alpha^{(b)}$ can be interpreted as the confidence in this member's prediction

- $\alpha^{(1)}, \alpha^{(2)}, \ldots \alpha^{(B)}$ are $> 0$

# AdaBoost Algorithm

Input : Training data $\mathcal{T} = \{ \underline{x}^{(i)}, y^{(i)} \}_{i=1}^{N}$

Output : 'B' weak classifiers

1) Assign weights $w_i^{(1)} = 1/N$ to all data points

2) for $b = 1, \ldots, B$ do

- Train a weak classifier $\hat{y}^{(b)}(\underline{x})$ on the weighted data $\{ \underline{x}^{(i)}, y^{(i)}, w_i^{(b)} \}_{i=1}^{N}$

- Compute $E_{train}^{(b)} = \sum_{i=1}^{N} w_i^{(b)} \, \mathbb{I}\{ y^{(i)} \neq \hat{y}^{(b)}(\underline{x}^{(i)}) \}$

- Compute $\alpha^{(b)} = 0.5 \ln \left( \dfrac{1 - E_{train}^{(b)}}{E_{train}^{(b)}} \right)$

- Compute $w_i^{(b+1)} = w_i^{(b)} \exp\left( -\alpha^{(b)} y^{(i)} \hat{y}^{(b)}(\underline{x}) \right)$

- Set $w_i^{(b+1)} \leftarrow w_i^{(b+1)} \Big/ \sum_{j=1}^{N} w_j^{(b+1)}$ for $i = 1, 2, \ldots, N$