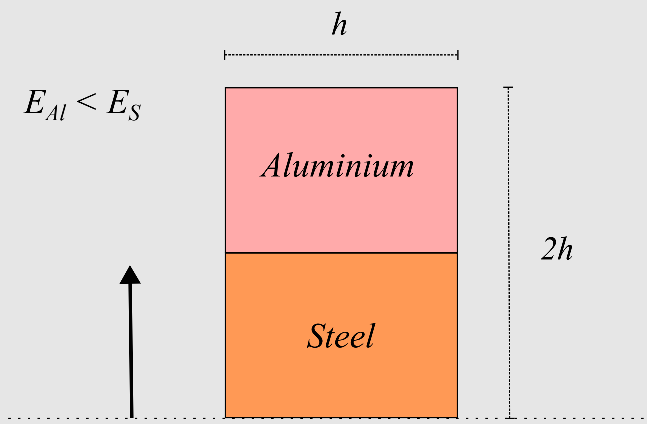
Tutorial 9 solution

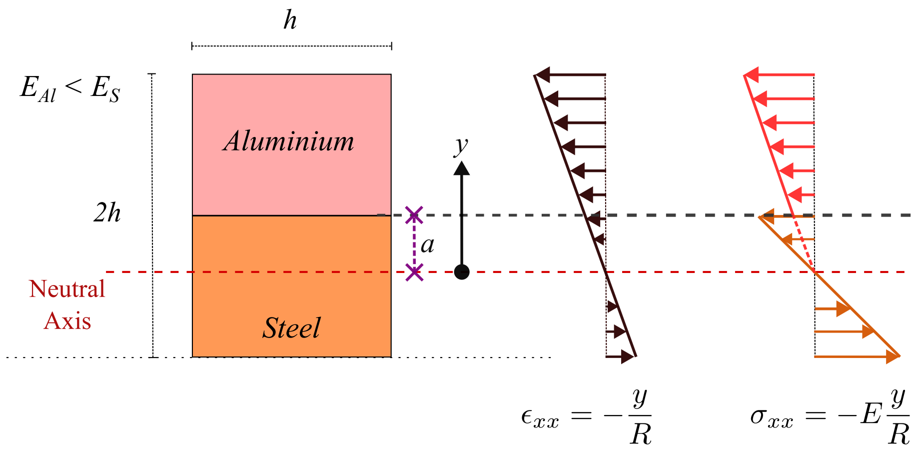
APL 104 - 2022 (Solid Mechanics)

Q1. Think of a composite beam having a rectangular cross-section such that one half of the cross-section (having a square shape) is aluminum while the other half is steel. When such a beam is bent, where will the neutral axis lie in the cross-section (calculated from the bottom line of the cross-section)?



Solution: Composite beams are constructed from more than one materials to increase their strength. In class, you have only seen beams made up of just one material. In general, the neutral axis of a composite beam will not lie at the geometric centroid of the cross-section.

Let us suppose that the neutral axis (NA) lies at a distance 'a' from the midline of the cross-section. The bending strain profile will be linear, continuous and passes through zero at the NA as shown below.



It obeys the following formula:

$$\epsilon_{xx} = -\frac{y}{R}$$

where y is the distance from neutral axis. However, the bending stress profile will be discontinuous at the location where the material changes. It follows the folowing formula:

$$\sigma_{xx} = E\epsilon_{xx} = \begin{cases} \sigma_{xx}^S = -E_S\frac{y}{R} & (-(h-a) < y < a) \\ \sigma_{xx}^A = -E_A\frac{y}{R}, & (a < y < h+a) \end{cases}$$

To obtain 'a', we use the fact that the total axial force must vanish in the cross-section (due to pure bending), i.e.,

$$\begin{aligned} &\int\int_{A_S} \sigma_{xx}^S dA + \int\int_{A_A} \sigma_{xx}^A dA = 0 \\ \Rightarrow\ & -\frac{E_S}{R}\int_{-(h-a)}^{a} ydy - \frac{E_A}{R}\int_{a}^{h+a} ydy = 0 \\ \Rightarrow\ & \left[(a)^2 - (h-a)^2\right] + \frac{E_A}{E_S}\left[(h+a)^2 - (a)^2\right] = 0 \\ \Rightarrow\ & -h^2\left(1 - \frac{E_A}{E_S}\right) + 2ha\left(1 + \frac{E_A}{E_S}\right) = 0 \\ \Rightarrow\ & a = \frac{h}{2}\left(\frac{E_S - E_A}{E_S + E_A}\right). \end{aligned}$$

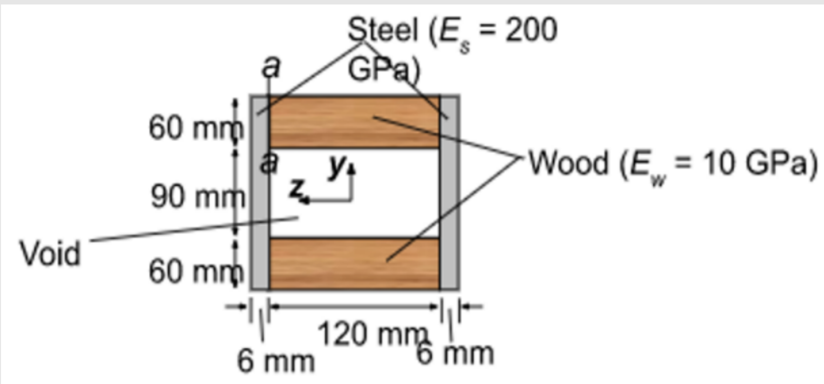
Note that when we set $E_s = E_A$ assuming both the materials to be the same, we indeed get $a = 0$ or the neutral axis then passes through the geometric center.

Q2. A beam of composite cross-section is subjected to bending moment $M_z = 30\text{kN}$. Find:

(a) The curvature induced in the beam

(b) Maximum bending stress in wood

(c) Maximum bending stress in steel



Solution:
We first obtain the bending stiffness of the cross-section. The cross-section being symmetrical, the neutral axis will be the mid horizontal line in the cross-section. The total bending

stiffness of the cross-section will simply be

$$
\begin{aligned}
(EI)_{tot} &= (EI)_{steel} + (EI)_{wood} \\
&= 2E_s \frac{1}{12} 0.006 \times 0.21^3 + E_w \frac{1}{12} \left(.12 \times .21^3 - .12 \times .09^3\right).
\end{aligned}
$$

(i) The curvature induced in the beam would then simply be

$$\kappa = M_z/(EI)_{tot}$$

(ii) Maximum bending stress in steel would be in its topmost/bottom-most fiber, i.e.,

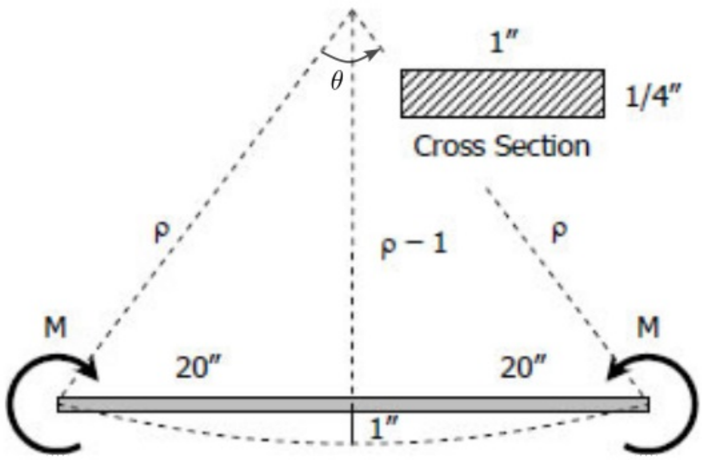
$$\sigma_s^{top} = E_s \kappa \, (0.045 + 0.060).$$

(iii) Likewise, the maximum bending stress in the wood fiber will be in its topmost/bottom-most fiber, i.e.,

$$\sigma_w^{top} = E_w \kappa \, (0.045 + 0.060).$$

> **Q3.** A flat steel bar, 1 inch wide by 0.25 inch thick and 40 inch long, is bent by couples applied at the ends so that the midpoint deflection is 1 inch. Compute the stress in the bar and the magnitude of the applied couples. Use $E = 200\text{GPa}$.

Solution:



Note that the deflection of the beam is very small compared to its length. Therefore, in the above picture, we have drawn the ends of the deformed beam to also coincide with the ends of the undeformed beam. In reality, deformed beam and undeformed beam will be of the same length due to pure bending deformation. However, the deformed beam being curved, its two ends will be slightly inward compared to the undeformed beam. Neglecting this mismatch, we can write based on geometry that

$$
\begin{aligned}
&(\rho - \delta)^2 + L^2/4 = \rho^2 \\
&\Rightarrow \rho^2 - 2\rho\delta + \delta^2 + L^2/4 = \rho^2 \\
&\text{or } \rho = \frac{L^2 + 4\delta^2}{8\delta} = 200.5 \text{ in.}
\end{aligned}
$$

Another way to obtain radius of curvature ρ is as follows. Let the angle subtended by the deformed beam at the center is $\theta/2$. So, $\rho\theta = L$. One can then write using trigonometry that

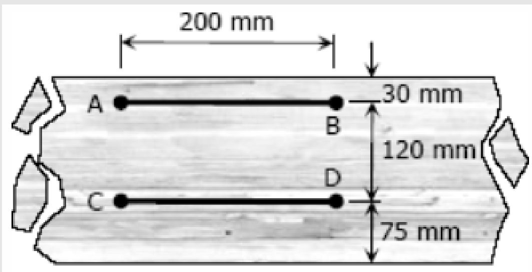
$$\begin{aligned}
&\rho\cos(\theta/2) = \rho - \delta \\
&\Rightarrow \rho(1 - \theta^2/8) = \rho - \delta \quad \text{(assuming } \theta \text{ to be very small)} \\
&\Rightarrow \rho\left(1 - \frac{L^2}{8\rho^2}\right) = \rho - 1 \\
&\text{or } \rho = \frac{L^2}{8\delta} = 200\ in.
\end{aligned}$$

With the above value of ρ, θ turns out to be 0.2 radian or approximately 11 degrees. This validates our assumption. The couple required to generate this bending will be

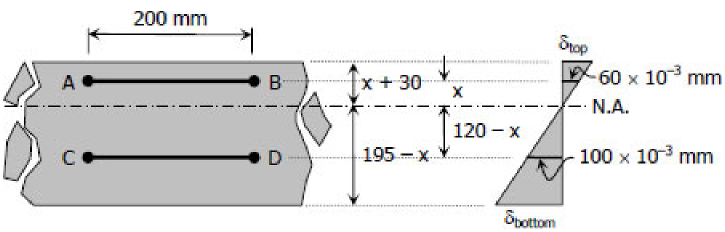
$$M = \frac{EI}{\rho} = \frac{(29 \times 10^6)\,\frac{1\,(1/4)^3}{12}}{200.5} = 188.3 \text{ lb.in (answer)}.$$

Here, the Young's modulus E has been converted into the units of lb/in^2.

Q4. In a laboratory test of a beam loaded by end couples, the longitudinal fibers at layer AB in the figure below are found to increase 60×10^{-3}mm whereas those at CD decrease $100 \times 10^{-3} mm$ in the 200mm-gauge length. Using $E = 70GPa$, determine the flexural stress in the top and bottom fibers.



Solution: The picture above is that of a section of the beam along its length. The shape of the cross-section is not mentioned here. Accordingly, we can not assume that the neutral axis lies at the center. Let it lie at a distance x below the longitudinal fiber AB (see the figure below).



We know that $\epsilon_{xx} = -\frac{y}{R}$. Hence $\frac{y}{\epsilon_{xx}}$ must be a constant, i.e.,

$$\begin{aligned}
\frac{x}{\epsilon_{AB}} &= \frac{x-120}{\epsilon_{CD}} \\
\Rightarrow \frac{x}{\frac{60\times10^{-3}}{200}} &= \frac{x-120}{-\frac{100\times10^{-3}}{200}} \\
\Rightarrow x &= 0.6\,(120 - x) \\
\text{or } x &= 45\text{mm}.
\end{aligned}$$

We now obtain flexural/bending stress in the top fiber. We can write the following for the bending strain in the top fiber:

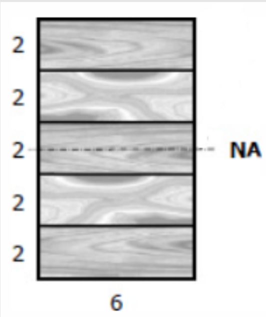
$$\frac{x}{\epsilon_{AB}} = \frac{x+30}{\epsilon_{top}} \Rightarrow \epsilon_{top} = \frac{x+30}{x}\epsilon_{AB} = \frac{75}{45}\frac{60\times10^{-3}}{200} = 5\times10^{-4}.$$

The bending stress in the top fiber will simply be $E\epsilon_{top}$ since the longitudinal fibers are under uniaxial loading during pure bending. Hence

$$\sigma_{top} = E\epsilon_{top} = 70\times10^{9}\ \times\ 5\times10^{-4} = 35\ MPa.$$

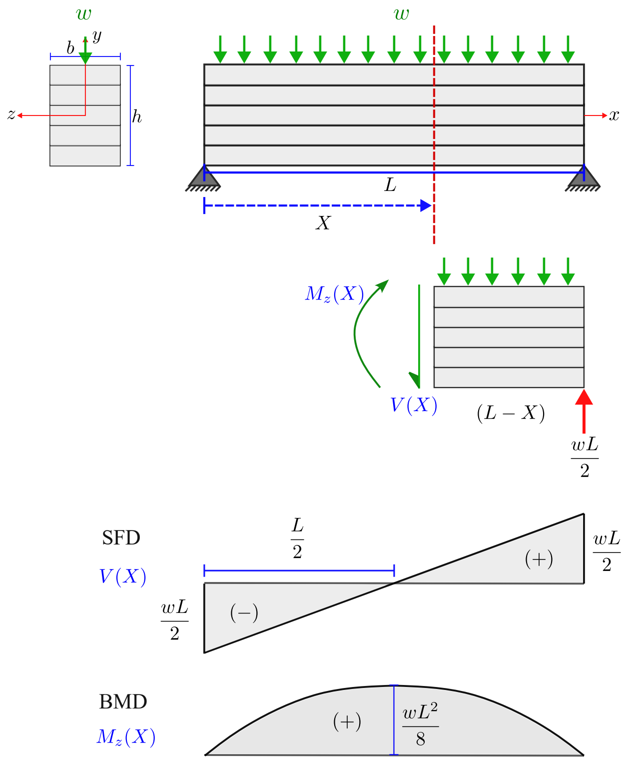
One can likewise obtain bending stress in the bottom-most fiber.

Q5. A laminated beam is composed of five planks, each 6 in. by 2 in. glued together to form a section 6 in. wide by 10 in. high.



The allowable shear stress in the glue is 90 psi, the allowable shear stress in the wood is 120 psi, and the allowable flexural stress in the wood is 1200 psi. Determine the maximum uniformly distributed load that can be carried by a simply supported beam on a 6ft simple span.

Solution: Assume that the uniformly-distributed load is applied along the y-axis. Note the rectangular cross-section is symmetrical about the y and z axes. Hence, they are also the principal axes.



- $\uparrow + \sum F_y \ = \ 0$

$$- V_x - w\left(L \ - \ x\right) + \frac{wl}{2} = 0$$

$$\Rightarrow V_x = -w\left(\frac{L}{2} - x\right)$$

- $\sum M_{\text{left end}} = 0$ (CCW sense +ve)

$$- M_z\left(x\right) + \frac{wl}{2}\left(L - x\right) - \frac{w\left(L - x\right)^2}{2} = 0$$

$$M_z\left(x\right) = \frac{wl}{2}\left(L - x\right) - \frac{w\left(L - x\right)^2}{2}$$

Since the load is applied along the principal axis, it will result in symmetrical bending. Also, the UDL will lead to both bending moment and shear force at any cross-section, hence, a non-uniform symmetrical bending would be observed.

To find the maximum flexural and shear stress, we need to find the bending and shear stress distribution. For symmetrical bending, the formula for bending and shear stresses is relatively simple:

$$\sigma_{xx} \ = \ \frac{M_z(x)\, y}{I_{zz}}, \quad \tau_{yz} \ = \ \frac{V(x)\, Q(y)}{I_{zz} b}$$

Note that if the load was applied along non-principal axes, it would have resulted in non-uniform unsymmetrical bending.

Distributions of $M_z(x)$ and $V(x)$ along the length of the beam are obtained from the bending moment diagram (BMD) and shear force diagram (SFD) as shown in the previous figure.

Max bending stress

$$\begin{aligned}
\sigma^{\text{wood}}_{xx,\max} &= \max_{x,y}\left\{\frac{M_z\left(x\right) y}{I_{zz}}\right\} \\
&= \max_{x} \frac{M_z\left(x\right)}{I_{zz}} \max_{y} y \\
&= \frac{\left(\dfrac{wL^2}{8}\right)\left(\dfrac{h}{2}\right)}{\dfrac{1}{12} b h^3} \\
&= \frac{3wL^2}{4bh^2} \ \leq \ \sigma^{\text{wood}}_{xx,\text{tol}} \quad \ldots \quad (1)
\end{aligned}$$

Max shear stress

$$\tau_{yx}(x,y) = \frac{V(x)Q(y)}{I_{zz}b}$$

$$\tau_{yx,\max} = \max_{x,y} \frac{V(x)\ Q(y)}{I_{zz}\ b} = \frac{1}{I_{zz}b} \max_x V(x) \quad \max_y Q(y)$$

$\rightarrow \max\limits_x V(x)$ occurs at $x = 0$ or $x = L$

$$\begin{aligned}
\rightarrow Q(y) &= \text{moment of area} \\
&= \text{area of the shaded region} \times \text{centroid of the shaded area from the NA} \\
&= b\left(\frac{h}{2} - y\right) \times \left\{ y + \frac{\frac{h}{2} - y}{2} \right\} \\
&= b\left(\frac{h}{2} - y\right) \times \frac{1}{2}\left(\frac{h}{2} + y\right) \\
&= \frac{b}{2}\left(\frac{h^2}{4} - y^2\right) \quad [Q(y) \text{ becomes maximum at } y = 0] \\
\rightarrow \max_y Q(y) &= \frac{b}{2}\frac{h^2}{4}
\end{aligned}$$

$$\therefore\ \tau_{yx,\max}^{\text{wood}} = \frac{V(x = L/2)}{I_{zz}} \frac{Q(y = 0)}{b} = \frac{\frac{wL}{2}\left(\frac{b}{2}\frac{h^2}{4}\right)}{\frac{1}{12}bh^3\ b} = \frac{3wL}{4bh} \leq \tau_{yx,\text{tol}}^{\text{wood}} \quad \dots \quad (2)$$

The glue between the wooden planks resists the shear stress generated between the planks. The shear stress at the level of the glue should also not exceed the maximum tolerable shear stress of the glue.

$$\therefore\ \tau_{yx,\max}^{\text{glue}} = \frac{V\left(x = \frac{L}{2}\right)}{I_{zz}} \frac{Q\left(y = \frac{h}{10}\right)}{b} = \frac{\frac{wL}{2}\left(\frac{3bh^2}{25}\right)}{\frac{1}{12}bh^3\ b} = \frac{18wL}{25bh} \leq \tau_{yx,\text{tol}}^{glue} \quad \dots \quad (3)$$

From Eqs. (1), (2), and (3), we would obtain three values of w. The maximum value of w should be the minimum of the values obtained from the three equations.

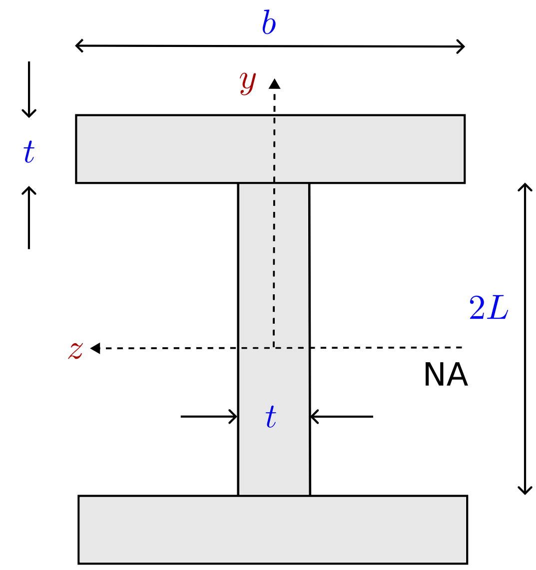
Q2. For an I-beam, assume the beam is subjected to a transverse load

(a) Obtain an expression for variation in shear stress τ_{xy} within its cross-section. You can use the formula $\tau_{xy} = \dfrac{VQ(y)}{I_{zz}b(y)}$.

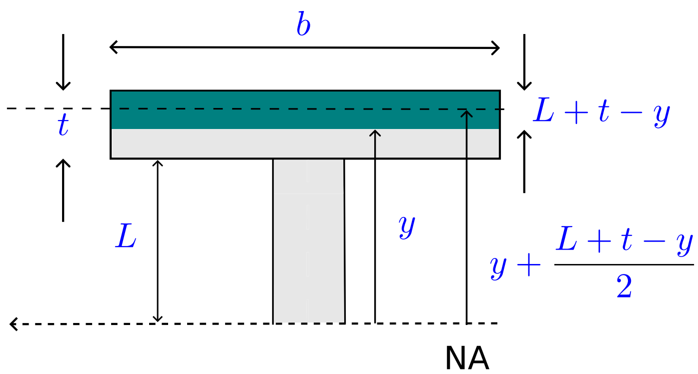
(b) Using the expression above, draw a graph depicting qualitative variation in shear stress within the cross-section.

(c) Where is the shear stress maximum? Find the ratio of maximum shear stress to average shear stress in the cross-section.

Solution: We will assume that the transverse load is acting along the principal axis of the beam, resulting in symmetrical bending. Therefore, we are instructed to use the simple formula $\tau_{yx}(x,y) = \dfrac{VQ(y)}{I_{zz}\, b(y)}$. Note that, unlike the beam with a rectangular cross-section, the width of the I-beam changes in the flange and the web. Hence, in this case, $b(y)$ in the denominator of the formula is taken to be a function of y.



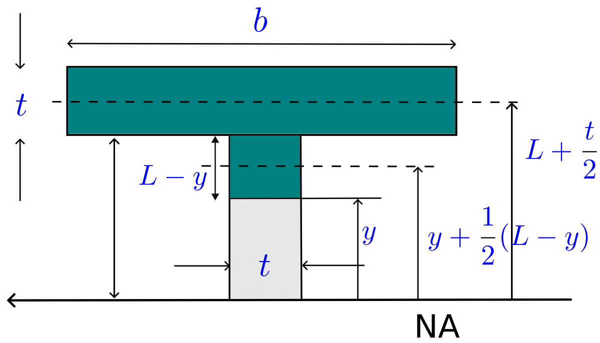
The shear force V is already given. Since we are looking at a section x, only $Q(y)$ and $b(y)$ will differ in the cross-section as a function of y. So let's find the distribution of $Q(y)$, which is the moment of the area. The Neutral Axis (NA) will be at the center aligned with the z-axis. It is clear from the figure of the I-beam that the geometry of the areas of the flange and the web is to be taken separately and will therefore lead to different distributions of shear stress.



<u>**Flange**</u>:

$$
\begin{aligned}
Q^f(y) &= b(t+L-y)\times\left[y+\frac{t+L-y}{2}\right] \\
&= \frac{b}{2}(t+L-y)(t+L+y) \\
&= \frac{b}{2}\left[(t+L)^2-y^2\right]
\end{aligned}
$$

$$\therefore\ \tau_{yx} = \frac{V_x}{I_{zz}}\frac{(b/2)}{b}\left[(t\,+\,L)^2\,-\,y^2\right] \quad (L \le y \le L + t)$$
$$= \frac{V_x}{2I_{zz}}\left[(t\,+\,L)^2\,-\,y^2\right]$$



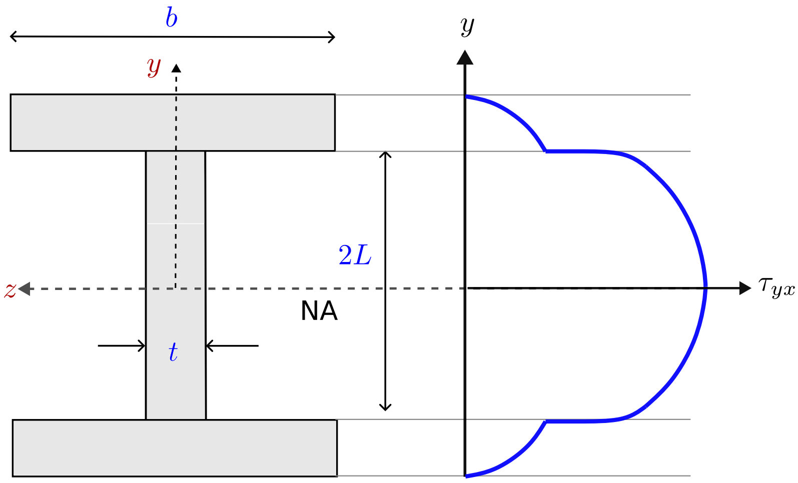
<u>**Web**</u>:

$$Q^w(y) = b\,t \times \left(L + \frac{t}{2}\right) + t\,(L-y) \times \frac{1}{2}\,(L+y)$$
$$= b\,t\left(L + \frac{t}{2}\right) + \frac{t}{2}\left(L^2 - y^2\right)$$
$$\therefore\ \tau_{yx} = \frac{V_x}{I_{zz}\,t}\left[b\,t\left(L + \frac{t}{2}\right) + \frac{t}{2}\left(L^2 - y^2\right)\right]$$

<u>I_{zz} for full I-beam</u>

$$I^f_{zz}\ \text{(for flange)} = \left[\frac{1}{12}bt^3\,+\,bt\left(L\,+\,\frac{t}{2}\right)^2\right]$$
$$I^w_{zz}\ \text{(for full web)} = \frac{1}{12} + (2L)^3$$
$$= \frac{8tL^3}{12}$$
$$I_{zz} = 2I^f_{zz}\,+\,I^w_{zz}$$

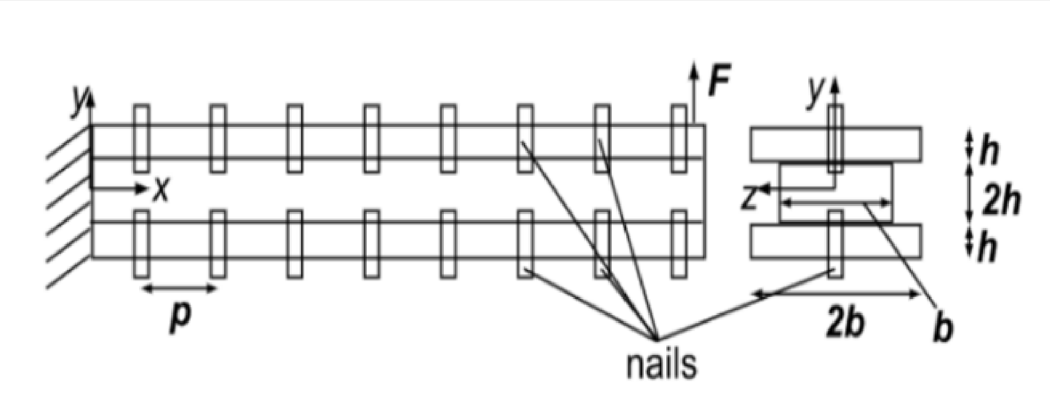
(b) The variation of shear stress with respect to y is shown below:



(c) Shear-stress is maximum at the NA

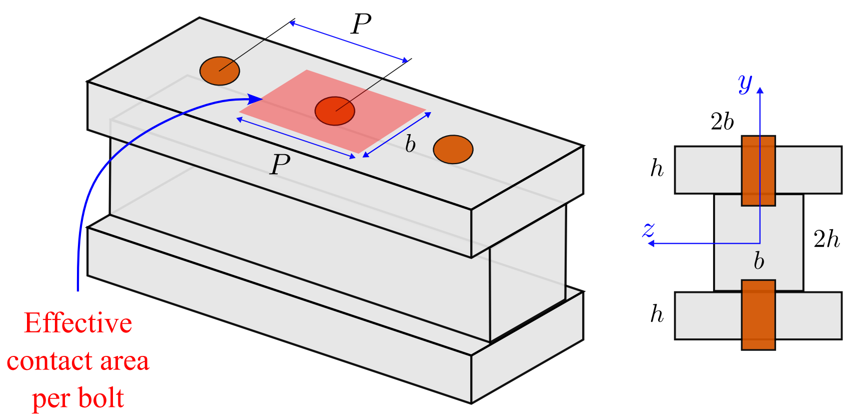
Avg shear stress, $\tau_{xy,\text{avg}} = \dfrac{V}{\text{Area of c/s of the I-beam}}$. Do the rest on your own.

Q3. A cantilever beam (figure below) is made up of three wooden planks nailed together as shown. Find the expression for shear force in nails in terms of F, p, b, and h. For the cross-section: $I_{zz} = 10bh^3$.



Solution: The shear stress at the contact surface between two planks is given by:

$$\tau_{yx}(x, y) = \frac{V(x)Q(y)}{I_{zz}\, b(y)}$$

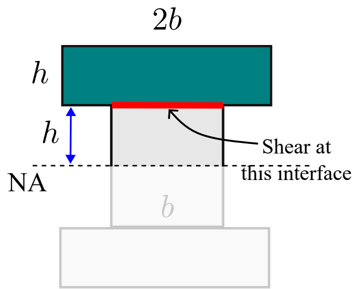


The effective area covered by each bolt has a length equal to the spacing between bolts. The total shearing force between the two planks must be resisted by the bolts.

$$\begin{aligned}\text{Resistive shear force per bolt} &= \tau_{yx} \times \text{Effective contact area} \\ &= \tau_{yx} \times (Pb)\ (b \text{ is taken as it is smaller out of the two width})\end{aligned}$$

We need to find the shear stress at the interface of the two planks. Therefore, $Q(y = h)$ will only have a contribution from the shaded region.

$$
\begin{aligned}
Q(y=h) &= \text{Moment of shaded area from NA} \\
&= \text{Area of shaded region} \\
&\qquad \times \text{centroid of that area from NA} \\
&= (2bh) \times \left(\frac{3h}{2}\right) \\
&= 3bh^2
\end{aligned}
$$



Assume that the shear force at any section is V, then

$$
\begin{aligned}
\tau_{yx} &= \frac{V\,(3bh^2)}{10bh^3 \times b} \\
&= \frac{3V}{10bh} \\
\text{Resistive shear force/bolt} &= \frac{3V}{10bh} \times Pb \\
&= \frac{3VP}{10h}
\end{aligned}
$$