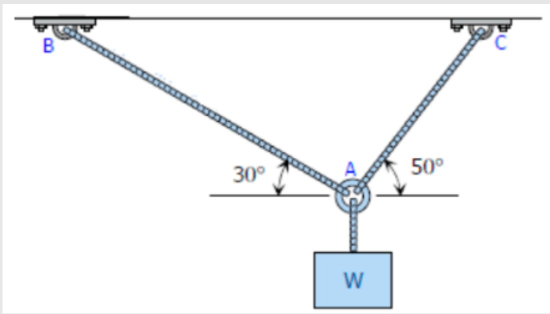
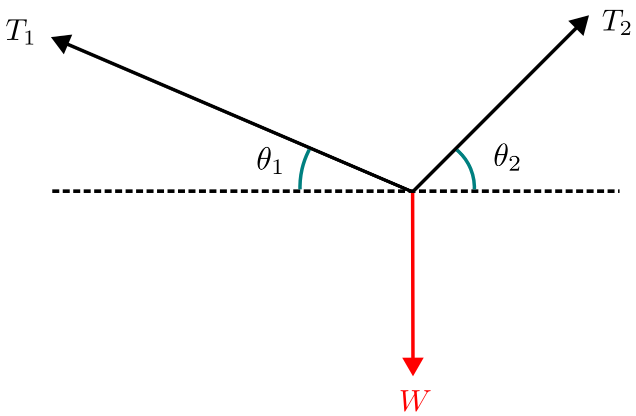
Tutorial 8 solution

APL 108 - 2025 (Mechanics of Solids)

Q1. Determine the largest weight W that can be supported by wire shown in figure shown below. The stress in either wire is not to exceed $207\text{N}/\text{mm}^2$. The cross-sectional area s of wires AB and AC are 258mm^2 and 323mm^2, respectively.



Solution:
Using free-body diagram and doing force balance, find tension in wires.



$$\overset{+}{\rightarrow} \sum F_x = 0 \Rightarrow T_2 \cos\theta_2 - T_1 \cos\theta_1 = 0$$
$$\Rightarrow T_2 = T_1 \frac{\cos\theta_1}{\cos\theta_2}$$

$$
\begin{aligned}
+\uparrow \sum F_y \;=\; 0 \Rightarrow\; & T_1 \sin\theta_1 \;+\; T_2 \sin\theta_2 \;=\; W \\
\Rightarrow & T_1 \sin\theta_1 \;+\; T_1 \frac{\cos\theta_1}{\cos\theta_2} \sin\theta_2 \;=\; W \\
\Rightarrow\; & T_1 \sin\theta_1 \;+\; T_1 \cos\theta_1 \tan\theta_2 \;=\; W \\
\Rightarrow\; & T_1 \;=\; \frac{W}{\cos\theta_1 \left(\tan\theta_1 \;+\; \tan\theta_2\right)} \\
\text{and } T_2 \;=\; & \frac{W}{\cos\theta_2 \left(\tan\theta_1 \;+\; \tan\theta_2\right)}
\end{aligned}
$$

The stress in each wire should not exceed the prescribed limit, so check that

$$
\begin{aligned}
\sigma_1 \;=\; \frac{T_1}{A_1} \;=\; & \frac{W}{A_1 \cos\theta_1 \left(\tan\theta_1 \;+\; \tan\theta_2\right)} \;\leq\; \sigma_{tol} \\
\Rightarrow\; W \;\leq\; & \sigma_{tol} \left[A_1 \cos\theta_1 \left(\tan\theta_1 \;+\; \tan\theta_2\right)\right] \qquad (1)
\end{aligned}
$$

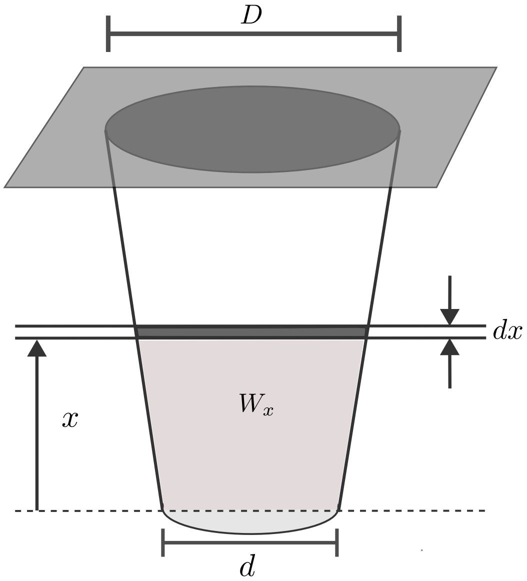
Similarly,

$$
\begin{aligned}
\sigma_2 \;=\; \frac{T_2}{A_2} \;=\; & \frac{W}{A_2 \cos\theta_2 \left(\tan\theta_1 \;+\; \tan\theta_2\right)} \;\leq\; \sigma_{tol} \\
\Rightarrow\; W \;\leq\; & \sigma_{tol} \left[A_2 \cos\theta_2 \left(\tan\theta_1 \;+\; \tan\theta_2\right)\right] \qquad (2)
\end{aligned}
$$

The maximum permissible load W would be the minimum of the loads obtained from relations 1 and 2. Note that as the wires undergo extension, the angles (θ_1, θ_2) would also change. However, we have assumed that such changes are negligible here.

Q2. A round bar of length L, which tapers uniformly from a diameter D at one end to a smaller diameter d at the other, is suspended vertically from the large end. If w is the weight per unit volume, find the elongation of the rod caused by its own weight.

Solution: Note that this problem represents a uniaxial loading scenario where the suspended bar is subjected to tension (due to its self-weight) along the vertical direction.



Note the downward force at any point in a given section is same. Hence, the elongation $d\Delta$ of an incremental section of thickness dx at a distance x from the bottom, is given by the formula:

$$d\Delta \;=\; \frac{F\,l}{AE} \;=\; \frac{W_x\,dx}{A_x E}$$

where W_x is weight of the shaded portion below the incremental section of thickness dx and A_x is the cross-sectional area of the incremental section both of which varying with x. The diameter of the relevant section and its area A_x can be found as

$$d_x \;=\; d \;+\; \frac{D\;-\;d}{L}x, \quad A_x \;=\; \frac{\pi\,d_x^2}{4}.$$

The weight $W_x = V_x \times w$ where V_x is the volume of the shaded truncated cone. This volume can be easily obtained as the sum of following two volumes: (a) volume of an internal cylinder of diameter d, and (b) volume of an outer cone of base diameter $d_x - d$, i.e.,

$$V_x = \frac{\pi}{4}d^2x + \frac{1}{3}\frac{\pi}{4}(d_x - d)^2x = \frac{\pi}{4}d^2x + \frac{1}{3}\frac{\pi}{4}\frac{(D-d)^2}{L^2}x^3$$

Accordingly, the downward weight W_x becomes

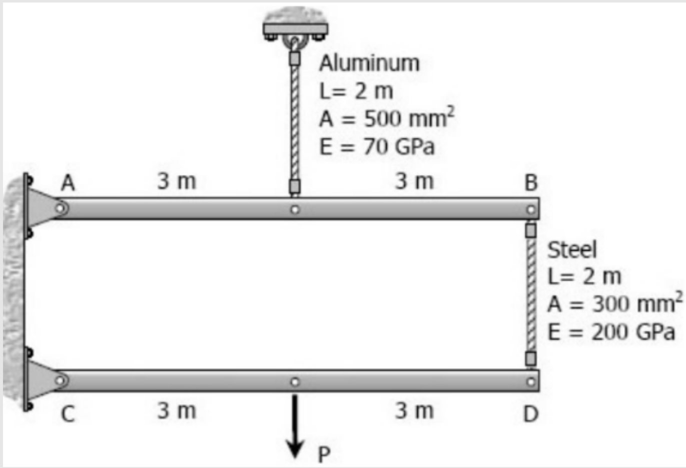
$$W_x = \left(\frac{\pi}{4}d^2x + \frac{1}{3}\frac{\pi}{4}\frac{(D-d)^2}{L^2}x^3\right) \times w$$

and the total elongation of the suspended bar is

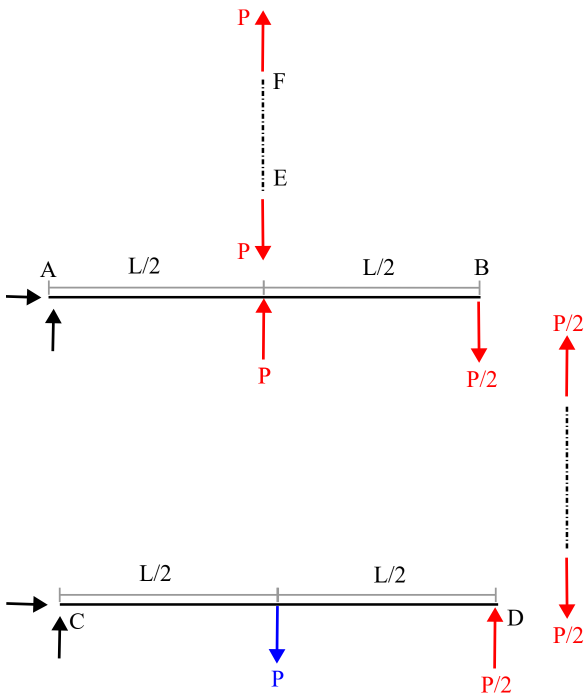
$$\delta L = \int_0^L \frac{W_x dx}{A_x E}$$

.

Q3. The rigid bars AB and CD shown in figure below are supported by pins at A and C and the two rods. Determine the maximum force P that can be applied as shown if its vertical movement is limited to 5mm. Neglect the wights of all members.



Solution: The bars AB and CD are rigid weightless bars. Since the aluminum and steel ropes are stretchable, they will deform and the assembly will undergo displacement.
First find the force acting in elements EF and BD to find how much they will stretch. To do that, we draw free diagrams and do force and moments balance for all members. Note that all joints are pinned joints an dhence no moment acts at these joints.



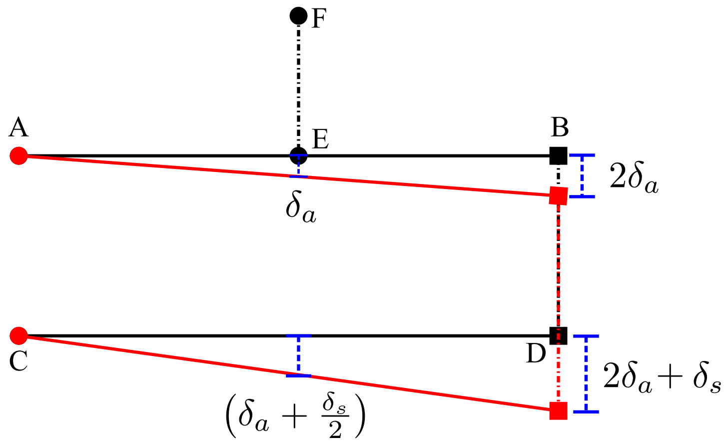
Next let's find the extensions of the ropes EF and BD.

aluminum rope EF:

$$\delta_a = \frac{Fl}{EA} = \frac{PL_a}{E_a A_a}$$

This implies point E goes down by δ_a whereas point B will accordingly go down by $2\delta_a$.
Steel rope BD:

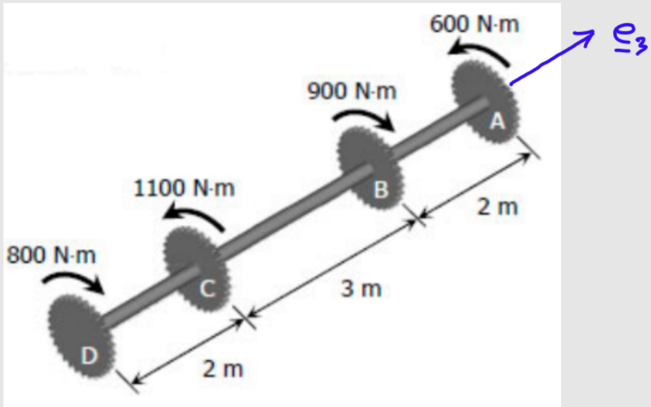
$$\delta_s = \frac{Fl}{EA} = \frac{(P/2)L_s}{E_s A_s}$$



This implies point D goes down by $2\delta_a + \delta_s$ and hence the the displacement of the point of application of load P turns out to be

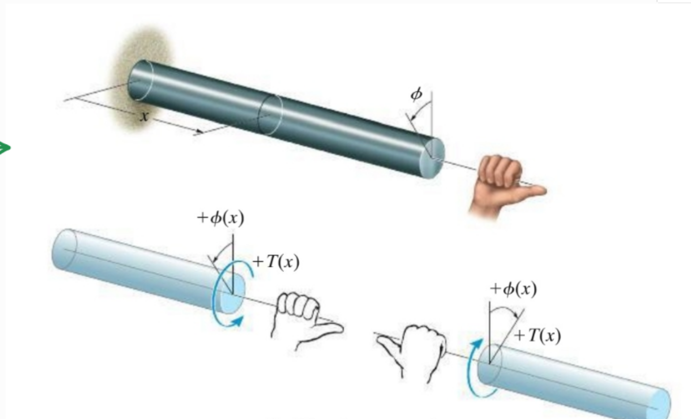
$$\begin{aligned}
\delta &= \delta_a + \frac{\delta_s}{2} \leq \delta_c \\
&= \frac{PL_a}{E_a A_a} + \frac{PL_s}{4E_s A_s} \leq \delta_c \\
\therefore P &\leq \frac{\delta_c}{\dfrac{L_{Al}}{E_{Al} A_{Al}} + \dfrac{L_s}{4E_s A_s}}
\end{aligned}$$

Q4. An aluminum shaft with a constant diameter of 50mm is loaded by torques applied to gears attached to its as shown in Fig. Using G = 28GPa, determine the relative angle of twist of gear relative to gear A.



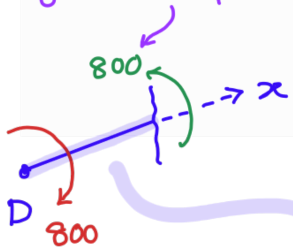
The shaft is in static equilibrium since the net torque = 0

Convention for torque diagram →

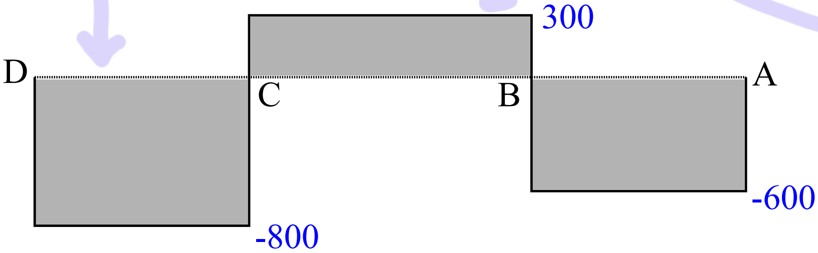


Positive sign convention for T and ϕ.

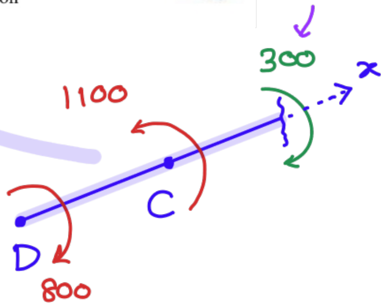
negative torque



Torque diagram



+ve torque



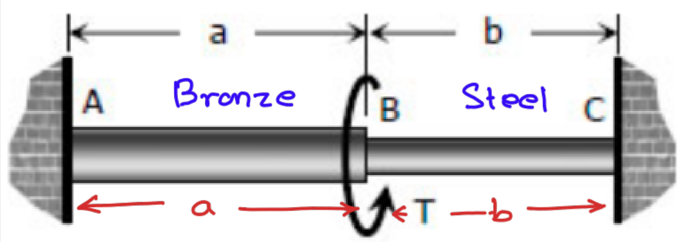
The relative rotation of gear D w.r.t. gear A is $\phi_{D|A} = \sum_{i=1}^{3} \frac{T_i L_i}{G_i J_i}$

$$\phi_{D|A} = \phi_{D|C} + \phi_{C|B} + \phi_{B|A}$$

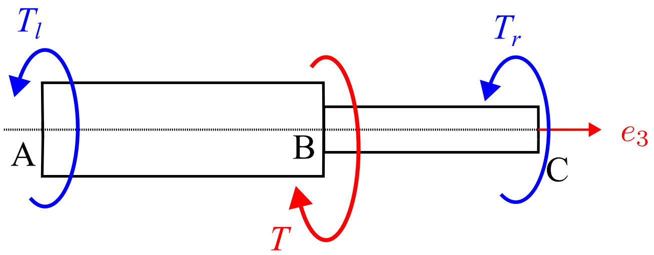
$$= \frac{1}{GJ}\left((-800)(2) + (300)(3) + (-600)(2)\right)$$

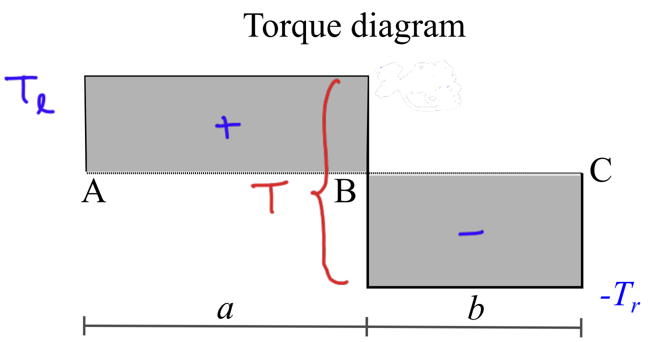
Finish the rest!

Q5. The compound shaft shown in the figure below is attached to rigid supports. For the bronze segment AB, the diameter is 75mm, $\tau_{\text{tol,B}} \leq 60$MPa, and $G = 35$GPa. For the steel segment BC, the diameter is 50mm, $\tau_{\text{tol,S}} \leq 80$MPa, and $G = 83$GPa. If $a = 2$m and $b = 1.5$m, compute the maximum torque T that can be applied.



Solution: Let the end reaction torques be T_l and T_r, respectively, at left and right fixed ends as shown in the figure below.





Equilibrium of the free-body gives us one equation along the e_3-direction

$$\overset{+}{\rightarrow}\!\!\!\!\rightarrow \sum M_3 = 0 \quad \Rightarrow \quad -T_l + T - T_r = 0$$

torque about e_3-axis

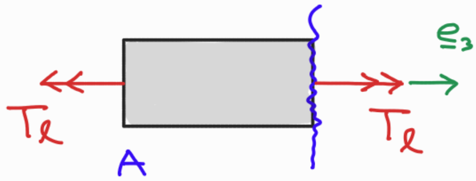
$$\Rightarrow \quad \boxed{T_r = T - T_l} \quad \text{—— } 1$$

Geometric compatibility requires that the ends of shafts do not undergo any twist as they are fixed. Hence, the relative twist of point C w.r.t point A must be zero.

$$\boxed{\phi_{C|A} = \phi_{C|B} + \phi_{B|A} = 0} \quad — (2)$$

Now, using load-displacement relation, $\phi = \frac{TL}{GJ}$, we can express the angle of twist in terms of torques T_l & T_r

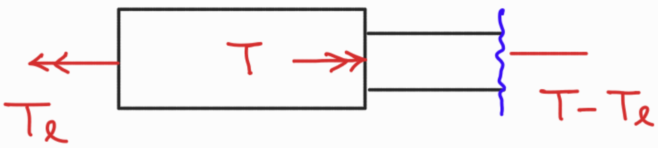
$$\phi_{C|B} + \phi_{B|A} = 0$$



T_l is +ve

$$\Rightarrow \frac{T_l L_{AB}}{G_B J_B} + \frac{-(T-T_l) L_{BC}}{G_S J_S} = 0$$

$$\Rightarrow \frac{T_l L_{AB}}{G_B J_B} = \frac{T_r L_{BC}}{G_S J_S}$$



$T-T_l$ is -ve

$$\Rightarrow \boxed{T_l = T_r \left[\frac{L_{BC}}{L_{AB}}\right]\left(\frac{G_B J_B}{G_S J_S}\right)} \quad — (3)$$

Substituting (3) in (1), we get:

$$T_r = T - T_r \left[\frac{L_{BC}}{L_{AB}}\right]\left(\frac{G_B J_B}{G_S J_S}\right) \Rightarrow T_r = \left(\frac{G_S J_S L_{BC}}{G_S J_S + G_B J_B}\right) T$$

$$\& \quad T_l = \left(\frac{G_B J_B L_{AB}}{G_S J_S + G_B J_B}\right) T$$

To determine the maximum torque T that can be applied, we have to check if the induced shear stresses lie within the limit of the permissible shear stresses for both steel and bronze.

Check for bronze

$$\tau_{B,max} = T_\ell \frac{R_B}{J_B} = \left(\frac{G_B J_B L_{AB}}{G_s J_s + G_B J_B}\right) T \frac{R_B}{J_B} < \tau_{B,tol}$$

$$\Rightarrow \quad T^{(1)} < \tau_{B,tol} \frac{J_B}{R_B} \left(\frac{G_s J_s + G_B J_B}{G_B J_B L_{AB}}\right)$$

Check for steel

$$\tau_{s,max} = T_r \frac{R_s}{J_s} = \left(\frac{G_s J_s L_{BC}}{G_s J_s + G_B J_B}\right) T \frac{R_s}{J_s} < \tau_{s,tol}$$

$$\Rightarrow \quad T^{(2)} < \tau_{s,tol} \frac{J_s}{R_s} \left(\frac{G_s J_s + G_B J_B}{G_s J_s L_{BC}}\right)$$

The maximum allowable torque T should be minimum of the two values

$$T = \min \{ T^{(1)}, T^{(2)} \}$$

Q6. The shaft shown in Figure 1 is made from a steel tube, which is bonded to a brass core. If a torque of $T = 250\text{N} \cdot \text{m}$ is applied at its end, plot the shear-stress distribution along a radial line of its cross-sectional area. Take $G_s = 80\text{GPa}$, $G_b = 36\text{GPa}$. (from Hibbeler, *Mechanics of Materials*, Ch 5)

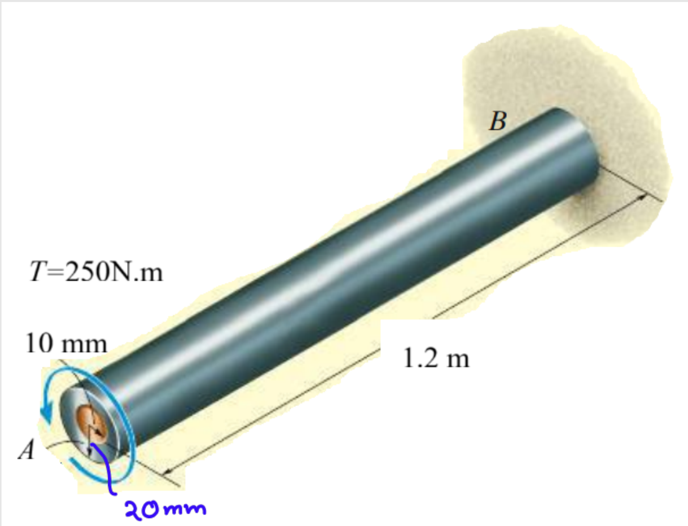
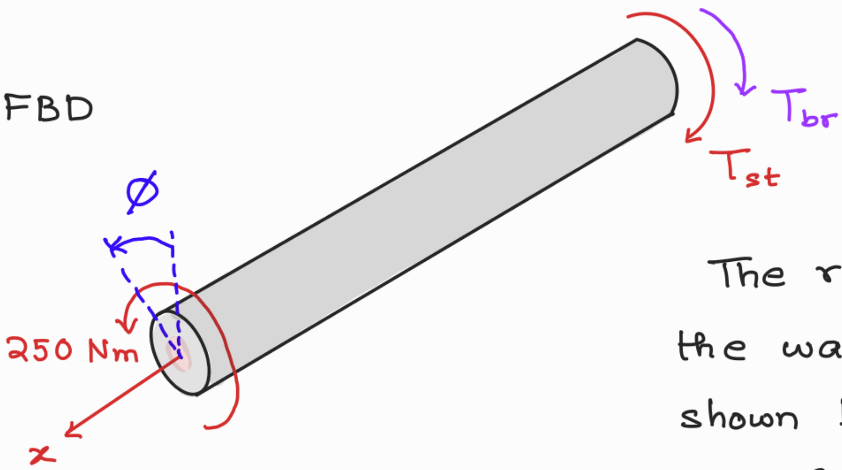


Figure 1: A composite shaft made up of a steel tube with a brass core.

Solution:

Draw FBD



The reaction at the wall has been shown by the unknown amt of torque resisted by the steel T_{st} and by the brass T_{br}

Equilibrium

$$\overset{+}{\swarrow} \sum M_x = 0 \quad \Rightarrow \quad \boxed{250 - T_{st} - T_{br} = 0} \quad \text{—} \quad (1)$$

Geometric compatibility requires the angle of twist of end A to be the same for both steel and brass since they are bonded together.

$$\boxed{\phi_{A/B} = \phi^{st}_{A/B} = \phi^{br}_{A/B}} \quad \text{---} \ (2)$$

Applying load-displacement relationship, $\phi = \dfrac{TL}{GJ}$

$$\phi^{st}_{A/B} = \phi^{br}_{A/B}$$

$$\Rightarrow \quad \frac{T_{st}\, \cancel{L}_{AB}}{G_{st}\, J_{st}} = \frac{T_{br}\, \cancel{L}_{AB}}{G_{br}\, J_{br}}$$

$$J_{st} = \frac{\pi}{2}\left(r_o^4 - r_i^4\right) = \frac{\pi}{2}\left(0.02^4 - 0.01^4\right)$$

$$J_{br} = \frac{\pi}{2} r_i^4 = \frac{\pi}{2}(0.01)^4$$

$$\Rightarrow \quad \boxed{T_{st} = 33.33\, T_{br}} \quad \text{---} \ (3)$$

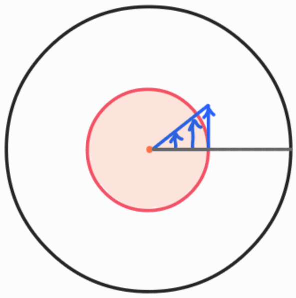
Using (3) in (1), we get

$$T_{st} = 242.72 \text{ Nm}$$

$$T_{br} = 7.282 \text{ Nm}$$

The shear stress in the brass core varies from zero at its center to a maximum at the interface where it contacts the steel tube. Using the torsion formula

$$\tau_{br,max} = \frac{T_{br}\, r_i}{J_{br}} = \frac{(7.282)\,(0.01)}{\frac{\pi}{2}\,(0.01)^4}$$



$$= 4.636 \times 10^6 \ N/m^2$$

For steel, the minimum and maximum shear stresses are

$$\tau_{st,min} = \frac{T_{st}\, r_i}{J_{st}} = \frac{(242.72)\,(0.01)}{\frac{\pi}{2}\,(0.02^4 - 0.01^4)} = 10.3 \times 10^6 \ N/m^2$$

$$\tau_{st,max} = \frac{T_{st}\, r_o}{J_{st}} = \frac{(242.72)\,(0.02)}{\frac{\pi}{2}\,(0.02^4 - 0.01^4)} = 20.6 \times 10^6 \ N/m^2$$

Note the discontinuity of shear stress at the brass-steel interface. This is expected, since the materials have diff. shear moduli, but note that shear strain is continuous at the interface (i.e. same for both brass and steel)