# APL 108 Tutorial 6 solutions

**Q1.** Think of the following displacement field in the body:

$$u_1 = 0.05x_1 + 0.03x_2^2,$$
$$u_2 = 0.07x_1x_2 + 0.08x_1^2,$$
$$u_3 = 0.$$

(a) Find the longitudinal strain of a line element along $\underline{e}_1$ direction at any point in the body.

(b) Determine the shear strain between line elements along $\underline{e}_1$ and $\underline{e}_2$

(c) Find volumetric strain for this displacement field. Does it vary from point to point?

(d) What is the shear strain between line elements along $\underline{e}_1$ and $\underline{e}_3$ at any point $(x_1, x_2)$?

(e) Determine the average local rigid-body rotation.

**Solution**:

$$[\nabla \underline{u}] = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$
$$= \begin{bmatrix} 0.05 & 0.06x_2 & 0 \\ 0.07x_2 + 0.16x_1 & 0.07x_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(a) Longitudinal strain along $\underline{e}_1$ direction

$$\epsilon_{nn} = \left(\underline{\underline{\epsilon}} \cdot \underline{n}\right) \cdot \underline{n}, \quad \underline{n} = \underline{e}_1$$
$$\epsilon_{11} = \frac{\partial u_1}{\partial x_1} = 0.05 \ \text{[ for all the points ]}$$

(b) Shear strain between line elements along $\underline{e}_1$ and $\underline{e}_2$: $\gamma_{12}$

$$\begin{aligned}\gamma_{12} &= 2\left(\underline{\underline{\epsilon}} \cdot \underline{n}\right) \cdot \underline{m}, \quad \underline{n} = \underline{e}_1, \underline{m} = \underline{e}_2 \\ &= \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \\ &= 0.06x_2 + 0.07x_2 + 0.16x_1 \\ &= 0.13x_2 + 0.16x_1\end{aligned}$$

(c) Volumetric strain: $\epsilon_v$

$$\begin{aligned}\epsilon_v &= \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \\ &= 0.05 + 0.07x_1 + 0 \\ &= 0.05 + 0.07x_1\end{aligned}$$

(changes from pt. to pt. if $x_1$ changes)

(d) Shear strain between line elements along $\underline{e}_1$ and $\underline{e}_3$: $\gamma_{13}$

$$\begin{aligned}\gamma_{13} &= \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \\ &= 0 + 0 \\ &= 0\end{aligned}$$

(e) Average local rigid-body rotation tensor is given by the anti-symmetric tensor $\underline{\underline{W}}$

$$\begin{aligned}\left[\underline{\underline{W}}\right] &= \frac{1}{2}\left([\nabla\, \underline{u}] - \left[\nabla\, \underline{u}^T\right]\right) \\ &= \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) & 0 \\ & 0 & 0 \\ \text{anti-sym} & & 0 \end{bmatrix}\end{aligned}$$

$$\begin{aligned}\frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) &= \frac{1}{2}(0.06x_2 - 0.07x_2 - 0.16x_1) \\ &= -0.005x_2 - 0.08x_1\end{aligned}$$

**Q2.** The displacement field for a body is given by

$$\underline{u} = k(x^2 + y)\hat{\imath} + k(y + z)\hat{\jmath} + k(x^2 + 2z^2)\hat{k}$$

Find the volumetric strain, shear strains $\gamma_{xy}$ and $\gamma_{yz}$, and the average local rotation tensor of the body at point $(2, 2, 3)$.

**Hint**:
Similar to the solution of Q1

**Q3.** The displacement gradient matrix at a point in a body is given by

$$\left[\underline{\underline{H}}\right] = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & 0 \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Write the condition for zero average local rotation.

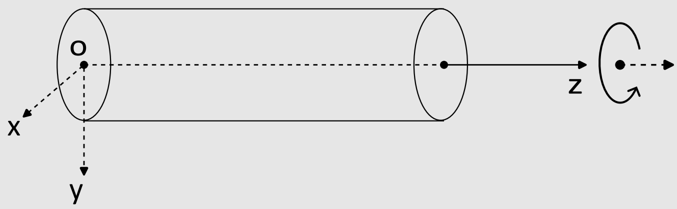
**Solution**:
For zero average local rotation, $\underline{\underline{W}} = \underline{\underline{0}}$
We are given $\underline{\nabla}\ \underline{u} = \underline{\underline{H}}$ from which $\underline{\underline{W}}$ can be derived as follows:

$$\left[\underline{\underline{W}}\right] = \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) \\ & 0 & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}\right) \\ \text{anti-sym} & & 0 \end{bmatrix}$$

All the components of the matrix are automatically zero except its (1,2) component and hence, to have zero infinitesimal rotation,

$$\frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) = 0 \Rightarrow \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}.$$

**Q4.** For a circular rod subjected to a torque (shown in figure below), the displacement components obtained at any point $(x, y, z)$ are as follows:




$$u_x = -\tau yz + ay + bz + c,$$
$$u_y = \tau xz - ax + ez + f,$$
$$u_z = -bx - ey + k$$

where $a$, $b$, $c$, $e$, $f$ and $k$ are constants and $\tau$ denotes twist.

(a) Select the constants $a$, $b$, $c$, $e$, $f$, $k$ such that the end section $z = 0$ is fixed in the following manner:

- Point $o$ has no displacement.
- The element $\Delta z$ of the axis rotates neither in the plane $xoz$ nor in the plane $yoz$
- The element $\Delta y$ of the axis does not rotate in the plane $xoy$.

(b) Determine the strain components.

(c) Verify whether these strain components satisfy the compatibility conditions.

**Solution**:

(a)

- Point 'o' has no displacement $\Rightarrow$ $u_x|_{(0,0,0)} = u_y|_{(0,0,0)} = u_z|_{(0,0,0)} = 0$

$$\Rightarrow \quad c = 0, \quad f = 0, \quad k = 0.$$

- All *local* rotations of line elements at the fixed end $z = 0$ are zero $\Rightarrow \underline{\underline{W}}\,|_{(0,0,0)} = \underline{\underline{0}}$

$$\left[\underline{\underline{W}}\right] = \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) \\ & 0 & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}\right) \\ \text{anti-sym} & & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{1}{2}\left[-\tau z + a - (\tau z - a)\right] & \frac{1}{2}\left[-\tau y + b - (-b)\right] \\ & 0 & \frac{1}{2}\left[\tau x + e - (-e)\right] \\ \text{anti-sym} & & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & (-\tau z + a) & \frac{1}{2}(-\tau y + 2b) \\ & 0 & \frac{1}{2}(\tau x + 2e) \\ \text{anti-sym} & & 0 \end{bmatrix}$$

$$\therefore \left[\underline{\underline{W}}\right]|_{(0,0,0)} = \begin{bmatrix} 0 & a & b \\ 0 & 0 & e \\ 0 & 0 & 0 \end{bmatrix}$$

Evaluating the above we get

$$a = 0,\ b = 0,\ e = 0.$$

(b) Strain components

$$\left[\underline{\underline{\epsilon}}\right] = \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) & \frac{\partial u_y}{\partial y} & \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) & \frac{\partial u_z}{\partial z} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & \frac{1}{2}(-\tau y) \\ & 0 & \frac{1}{2}(\tau x) \\ \text{sym} & & 0 \end{bmatrix}.$$

Note that $\gamma_{12} = 0$: the line elements in the cross-section undergo just rigid rotation during twist. Of course, different cross-sections rotate by different amounts, i.e., $\tau z$ (see part (a)).

(c) Strain compatibility is naturally satisfied since we derived the strain components from displacement functions. It is only when strain components are directly prescribed that we need to check for strain compatibility.

**Q5.** For the displacement field $u_x = k(x^2 + 2z)$, $u_y = k(4x + 2y^2 + z)$, $u_z = 4kz^2$ with $k = 0.001$, determine the change in angle between two lines segments PQ and PR at $P(2, 2, 3)$ having direction cosines before deformation as follows:

PQ: $n_{x1} = 0,\ \ n_{y1} = n_{z1} = \frac{1}{\sqrt{2}}$
PR: $n_{x2} = 1,\ \ n_{y2} = n_{z2} = 0$

**Hint**: Very similar to Q1 and Q2, only the direction of line elements are different

$$\gamma_{PQ,PR} = \left(\underline{\underline{\epsilon}}\ \underline{n}_{PR}\right) \cdot \underline{n}_{PQ} \text{ and then evaluate at P(2,2,3)}$$

**Q6.** Verify whether the following strain field satisfies the equations of compatibility. Here $p$ is a constant.

$$\epsilon_{xx} = py, \quad \epsilon_{yy} = px, \quad \epsilon_{zz} = 2p(x+y)$$

$$\gamma_{xy} = p(x+y), \;\; \epsilon_{yz} = 2pz, \;\; \epsilon_{zx} = 2pz$$

**Solution:** The six strain compatibility conditions are

$$\begin{aligned}
\frac{\partial^2 \epsilon_{xx}}{\partial y^2} + \frac{\partial^2 \epsilon_{yy}}{\partial x^2} &= \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}, \\
\frac{\partial^2 \epsilon_{yy}}{\partial z^2} + \frac{\partial^2 \epsilon_{zz}}{\partial y^2} &= \frac{\partial^2 \gamma_{yz}}{\partial y \partial z}, \\
\frac{\partial^2 \epsilon_{xx}}{\partial z^2} + \frac{\partial^2 \epsilon_{zz}}{\partial x^2} &= \frac{\partial^2 \gamma_{xz}}{\partial x \partial z}, \\
\frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right) &= 2\frac{\partial^2 \epsilon_{zz}}{\partial x \partial y}, \\
\frac{\partial}{\partial x}\left(\frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{xz}}{\partial y} - \frac{\partial \gamma_{yz}}{\partial x}\right) &= 2\frac{\partial^2 \epsilon_{xx}}{\partial x \partial z}, \\
\frac{\partial}{\partial y}\left(\frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{xz}}{\partial y}\right) &= 2\frac{\partial^2 \epsilon_{yy}}{\partial x \partial z}.
\end{aligned}$$

Notice that the strain compatibility equations involve second derivative of each of the strain components. As all the prescribed strain component functions are linear in $(x, y, z)$, their second derivartives will automatically vanish. Hence, the six equations are trivially satisfied.

**Q7.** Given the following formulas for strain components:

$$\epsilon_{xx} = 5 + x^2 + y^2 + x^4 + y^4,$$
$$\epsilon_{yy} = 6 + 3x^2 + 3y^2 + x^4 + y^4,$$
$$\gamma_{xy} = 10 + 4xy(x^2 + y^2 + 2),$$
$$\epsilon_{zz} = \gamma_{yz} = \gamma_{zx} = 0.$$

(a) Determine whether the above strain field is possible. If it is possible, determine the displacement components in terms of $x$ and $y$. Assume that $u_x = u_y = 0$ and $\omega_{xy} = 0$ at the origin.

(b) For the state of strain given in previous problem, write down the spherical and deviatoric parts and also determine the volumetric strain.

**Solution**: It can be observed that the above strain field represents a plane strain condition, where

$$\epsilon_{xx} = 5 + x^2 + y^2 + x^4 + y^4, \quad (1)$$
$$\epsilon_{yy} = 6 + 3x^2 + 3y^2 + x^4 + y^4, \quad (2)$$
$$\gamma_{xy} = 10 + 4xy\left(x^2 + y^2 + 2\right), \quad (3)$$

$$\epsilon_{zz} = \gamma_{yz} = \gamma_{zx} = 0 \text{ (all strains along z-direction is zero)}$$

(a) As this is a plane-strain case, only one out of the six strain compatibility condition need to be checked, other five are trivially satisfied. The condition to be checked is

$$\frac{\partial^2 \epsilon_{xx}}{\partial y^2} + \frac{\partial^2 \epsilon_{yy}}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}. \quad (4)$$

Using Eq. (1) and Eq. (2),

$$\frac{\partial^2 \epsilon_{xx}}{\partial y^2} + \frac{\partial^2 \epsilon_{yy}}{\partial x^2} = \left(2 + 12y^2\right) + \left(6 + 12x^2\right) = 12x^2 + 12y^2 + 8 \quad (5)$$

and using Eq. (3),

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = 12x^2 + 12y^2 + 8 \quad (6)$$

As Eq. (5) and Eq. (6) are equal, the strain fields are physically realizable.

To determine the displacement components $u_x$ and $u_y$, one can integrate the strains $\epsilon_{xx}$ and $\epsilon_{yy}$, respectively, and then determine the integration constants using shear strain relation and the prescribed boundary conditions.

$$\epsilon_{xx} = \frac{\partial u_x}{\partial x} \Rightarrow u_x = \int \epsilon_{xx} dx = 5x + \frac{x^3}{3} + y^2 x + \frac{x^5}{5} + xy^4 + f(y)$$

Similarly,

$$\epsilon_{yy} = \frac{\partial u_y}{\partial y} \Rightarrow u_y = \int \epsilon_{yy} dy = 6y + 3x^2 y + y^3 + x^4 y + \frac{y^5}{5} + g(x)$$

Note, $f(y)$ and $g(x)$ are not functions of $z$ because we are dealing with plane-strain case: $u_x$ and $u_y$ in such a case are independent of $z$. Let's now obtain $\gamma_{xy}$ from $u_x$ and $u_y$:

$$\begin{aligned}\gamma_{xy} &= \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \\ &= 2xy \;+\; 4xy^3 \;+\; f'(y) \;+\; 6xy + 4x^3y \;+\; g'(x)\end{aligned} \tag{7}$$

Upon comparing it with the given $\gamma_{xy}$ from Eq. (3), we get

$$\begin{aligned} f'(y) \;+\; g'(x) &= 10 \\ \Rightarrow f'(y) &= 10 - g'(x)\end{aligned} \tag{8}$$

The LHS is a function of $y$ whereas the RHS is a function of $x$ which can be true only when they are both constants, i.e.,

$$\begin{aligned} f'(y) &= 10 - g'(x) = d \\ \Rightarrow f(y) = c_1 y + c_2, &\text{ and } g(x) = (10-d)x + c_3.\end{aligned}$$

In order to obtain the constants $d$, $c_1$, $c_2$, and $c_3$, let us use the prescribed boundary conditions. Using the displacement BC at origin:

$$\begin{aligned} u_x|_{origin} &= 0 \;\Rightarrow\; f(0) \;=\; 0 \Rightarrow c_2 = 0, \\ u_y|_{origin} &= 0 \;\Rightarrow\; g\,(0) \;=\; 0 \Rightarrow c_3 = 0, \\ w_{xy}|_{origin} &= 0 \;\Rightarrow\; \frac{1}{2}\left(\frac{\partial u_x}{\partial y} \;-\; \frac{\partial u_y}{\partial x}\right)|_{origin} \;=\; 0 \\ &\Rightarrow\; f'(y) \;-\; g'(x)\,|_{origin} \;=\; 0 \\ &\Rightarrow\; f'(0) \;-\; g'(0) \;=\; 0 \\ &\Rightarrow\; d - (10-d) = 0 \\ &\Rightarrow\; d = 5.\end{aligned}$$

We thus have

$$f(y) \;=\; 5y, \text{ and } g(x) \;=\; 5x$$

Therefore the displacement components are:

$$\begin{aligned} u_x &= 5x \;+\; \frac{x^3}{3} \;+\; y^2x \;+\; \frac{x^5}{5} \;+\; xy^4 \;+\; 5y, \\ u_y &= 6y \;+\; 3x^2y \;+\; y^3 \;+\; x^4y \;+\; \frac{y^5}{5} \;+\; 5x.\end{aligned}$$

(b) • Spherical (or hydrostatic) strain tensor $= \frac{1}{3}J_1\underline{\underline{I}}$ where

$$\begin{aligned} J_1 &= \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz} \\ &= \left(5 + x^2 + y^2 + x^4 + y^4\right) + \left(6 + 3x^2 + 3y^2 + x^4 + y^4\right) + 0 \\ &= 11 + 4x^2 + 4y^2 + 2x^4 + 2y^4.\end{aligned}$$

- Deviatoric strain tensor $= \underline{\underline{\epsilon}} - \frac{1}{3} J_1 \underline{\underline{I}}$ and its matrix representation is

$$\begin{bmatrix} d_1 & \gamma_{xy}/2 & 0 \\ & d_2 & 0 \\ \text{sym} & & 0 \end{bmatrix}$$

where $d_1 = \left(5 + x^2 + y^2 + x^4 + y^4\right) - \dfrac{1}{3}\left(11 + 4x^2 + 4y^2 + 2x^4 + 2y^4\right)$,

$$d_2 = \left(6 + 3x^2 + 3y^2 + x^4 + y^4\right) - \frac{1}{3}\left(11 + 4x^2 + 4y^2 + 2x^4 + 2y^4\right).$$

- Volumetric strain $\epsilon_v = \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz} = J_1$.