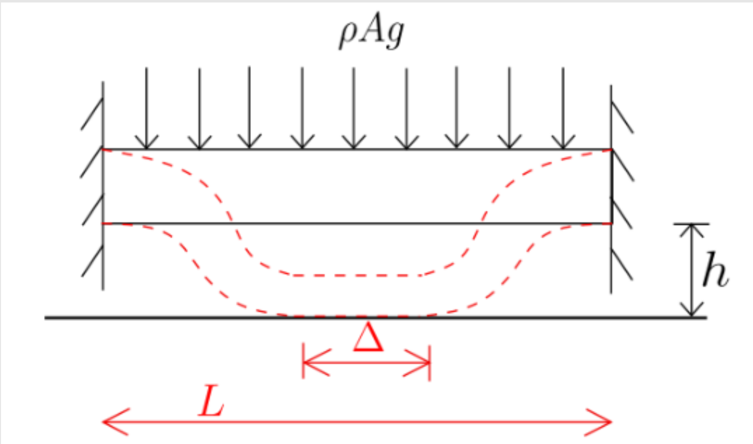
APL 108 Tutorial 10 solutions

Q1. Consider a beam clamped at both ends. The beam sags down due to its own weight as shown in Figure 10, the distributed weight being ρAg. However, the ground position (h below the beam) is such that some part of the beam rests on the ground upon deformation while the remaining part just hangs. Find the length of the beam Δ which will rest on the ground.



Solution: Notice that the problem is symmetrical with respect to the center of the beam ($x = \frac{L}{2}$). Let us consider the left hanging part of the beam, i.e., from $x = 0$ to $x = \frac{L-\Delta}{2}$. We draw its free body diagram as shown in Figure 1. Bending moment $M(0)$

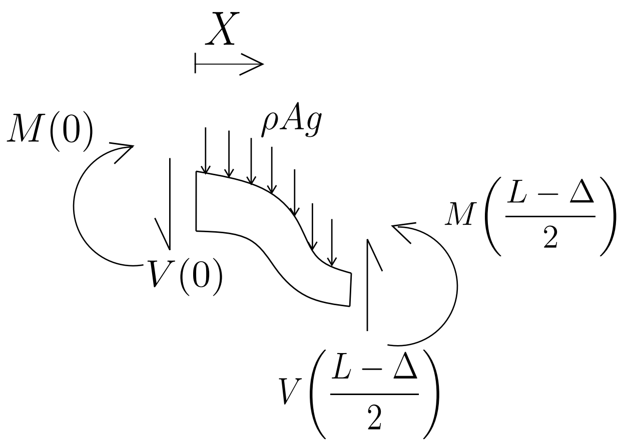


Figure 1: Free body diagram of the left hanging part of the beam.

and shear force $V(0)$ act on its left-end cross-section (applied by the clamped end). A shear force and a bending moment also act on the right end of this part (applied by the remaining part of the beam). All of these four quantities are unknown. To proceed with the Euler-Bernoulli beam equation

$$EI\frac{d^2v}{dx^2} = M(x) \tag{1}$$

we need to first find the bending moment profile $M(x)$. We thus cut a section at a distance x from the left end and draw the free body diagram of the right part as shown in Figure 2. Shear force and bending moment act on left and right-ends whereas

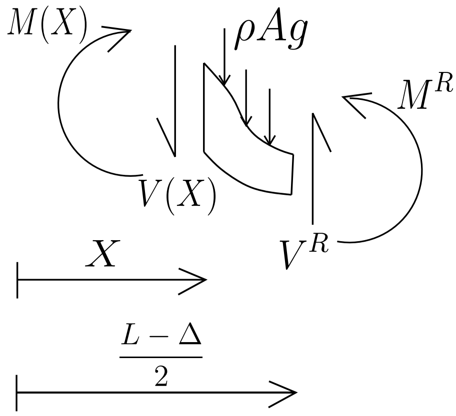


Figure 2: Free body diagram of a section of the hanging part of the beam at a distance x from the left end.

distributed load acts throughout its length. Its force balance gives us

$$V(x) = V^R - \rho Ag\left(\frac{L-\Delta}{2} - x\right) \tag{2}$$

We then do moment balance about the centroid of its left-end cross-section which yields

$$-M(x) + M^R + V^R\left(\frac{L-\Delta}{2} - x\right) - \frac{\rho Ag}{2}\left(\frac{L-\Delta}{2} - x\right)^2 = 0$$

$$\implies M(x) = M^R + V^R\left(\frac{L-\Delta}{2} - x\right) - \frac{\rho Ag}{2}\left(\frac{L-\Delta}{2} - x\right)^2 \tag{3}$$

Plugging this in equation (1), we get

$$\frac{d^2v}{dx^2} = \frac{1}{EI}\left(M^R + V^R\left(\frac{L-\Delta}{2} - x\right) - \frac{\rho Ag}{2}\left(\frac{L-\Delta}{2} - x\right)^2\right). \tag{4}$$

We now think of boundary conditions. As there are three additional unknown parameters (V^R, M^R, Δ), a total of five boundary conditions will be required. Out of these, two can be obtained as earlier from the clamped end at $x = 0$, i.e.,

$$y(0) = 0, \quad \frac{dv}{dx}(0) = 0 \tag{5}$$

We cannot use the clamped boundary condition of the original beam at $x = L$ because we are only analyzing the left-hanging portion of the full beam. If we carefully observe, we can see that at $x = \frac{L-\Delta}{2}$, the beam starts to come into contact with the ground surface. Thus, the deflection of this point is $-h$, i.e.,

$$v\left(\frac{L-\Delta}{2}\right) = -h \tag{6}$$

Also, as the ground is flat, the beam's deflection is uniform throughout the resting portion of the beam. Thus, the first and second derivatives of deflection in the resting portion will be zero which by continuity will also be zero at $x = \frac{L-\Delta}{2}$. Thus, we get our fourth boundary condition as

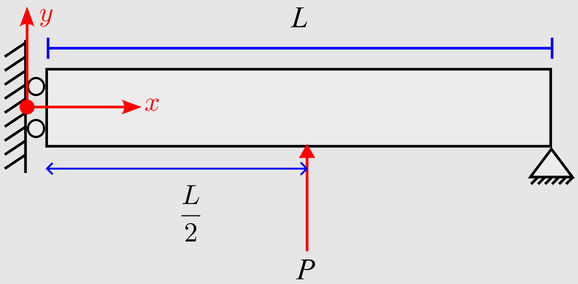
$$\frac{dv}{dx}\left(x = \frac{L-\Delta}{2}\right) = 0. \tag{7}$$

As the second derivative $\frac{d^2v}{dx^2}$ vanishes throughout in the resting region, the bending curvature and hence the bending moment will be zero in the entire resting region of the beam. Thus, M^R which is the bending moment at the edge of the flat region must also be zero.[1] This gives us the final boundary condition, i.e.,

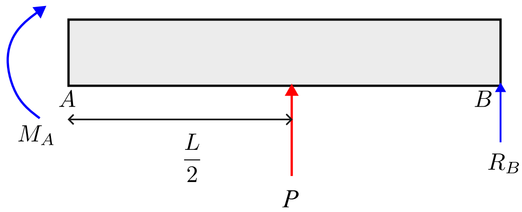
$$M^R = 0. \tag{8}$$

Using all the five boundary conditions in the EBT equation, we can solve for the deflection of the beam and also the length of the resting portion of the beam, i.e., Δ.

Q2. Suppose a beam is kept with roller support at one end ($x = 0$) constrained to only move in y-direction and pinned at the other end ($x = L$) as shown. The beam is subjected to transverse load (P) at the middle of the beam. Find the deflection of the beam using the Euler-Bernoulli beam theory.



Solution: The constraint at the left end has frictionless rollers which allows free translation in the vertical direction but no rotation. So there is a reaction moment at the left end but no vertical reaction force. The right end is sitting on a pin support which allows free rotation but does not allow translation. Thus, at the right end, the reaction moment is zero but the vertical reaction force is non-zero. The free-body diagram of the full beam thus looks as follows:



[1]There can also be no jump discontinuity in bending moment at $x = \frac{L-\Delta}{2}$ because of line contact of the beam with the ground due to which the ground does not exert any reactive bending moment at this point.

Since there are two unknown reactions in this case, we can determine them using the two static equilibrium equations, i.e., by doing force balance in y-direction and moment balance about (say) the left end.

$$+\uparrow \sum F_y = 0 \;\Rightarrow R_B = -P$$

$$\circlearrowleft\!\!\oplus \sum M_{\text{left end}} = 0 \;\Rightarrow -M_A + P\,\frac{L}{2} + \overset{-P}{\cancel{R_B}}\,L = 0 \Rightarrow M_A = -\frac{PL}{2}$$

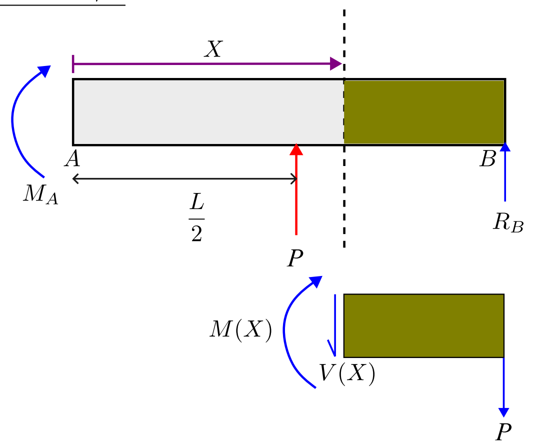
The reaction force R_B is $-$ve meaning it should act in the downward direction, and the moment M_A is $-$ve meaning that it should act in an anti-clockwise sense (opposite to what is shown in the free-body diagram).

Consider the origin of the coordinate system to be at the left end A. The deflection equation for Euler-Bernoulli beam theory is given by

$$EI\frac{d^2v}{dx^2} = M\,(x)$$

To use this formula, we need to find the expression for $M\,(x)$. Note that $M(x)$ in the cross section will have different expressions for $x < \frac{L}{2}$ and $x \geq \frac{L}{2}$. So, we will consider the two cases separately:

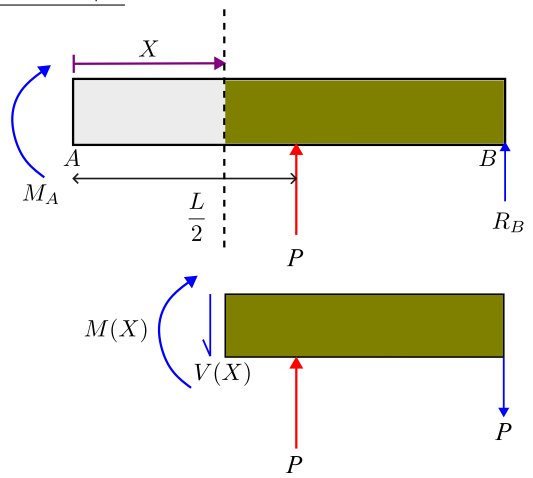
<u>For $x \geq L/2$:</u>



$$+\uparrow \sum F_y = 0 \;\Rightarrow\; V\,(x) = -P$$

$$\circlearrowleft\!\!\oplus \sum M_{\text{left end}} = 0$$
$$\Rightarrow\; -M\,(x) - P\,(L - x) = 0$$
$$\Rightarrow\; M\,(x) = -P\,(L - x)$$

<u>For $x < L/2$:</u>



$$+\uparrow \sum F_y = 0 \;\Rightarrow\; V\,(x) = -P + P = 0$$

$$\circlearrowleft\!\!\oplus \sum M_{\text{left end}} = 0$$
$$\Rightarrow\; -M\,(x) + P\left(\frac{L}{2} - x\right) - P\,(L - x) = 0$$
$$\Rightarrow\; M(x) = -\frac{PL}{2}$$

The boundary conditions for solving the problem are:

(a) **BC 1**: For $x < \dfrac{L}{2}$: Slope at $x = 0$ is zero, $\dfrac{dv}{dx}(0) = 0$

(b) **BC 2**: For $x \geq \dfrac{L}{2}$: Displacement at $x = L$ is zero, $v(L) = 0$

Let us find the deformation for each case separately:

- For $x < L/2$:

$$
\begin{aligned}
EI\frac{d^2v}{dx^2} &= \frac{-PL}{2} \\
\Rightarrow\ EI\frac{dv}{dx} &= \frac{-PL}{2}x + C_1 \\
\text{Use } \mathbf{BC\ 1}\text{: } \frac{dv}{dx}(0) &= 0,\ \text{we get } C_1 = 0 \\
\Rightarrow\ EI\frac{dv}{dx} &= \frac{-PL}{2}x \qquad (9) \\
\Rightarrow\ v(x) &= \frac{-PLx^2}{4EI} + C_2 \qquad (10)
\end{aligned}
$$

- For $x \geq L/2$:

$$
\begin{aligned}
M(x) &= -P(L-x) \\
EI\frac{d^2v}{dx^2} &= -P(L-x) \\
\Rightarrow\ EI\frac{dv}{dx} &= -PLx + \frac{Px^2}{2} + D_1 \qquad (11) \\
\Rightarrow\ v(x) &= \frac{1}{EI}\left(-\frac{PLx^2}{2} + \frac{Px^3}{6} + D_1x + D_2\right) \qquad (12)
\end{aligned}
$$

Here, we can find D_1 using continuity of slope at $x = \dfrac{L}{2}$

$$
\begin{aligned}
\frac{dv}{dx}\left(x = \frac{L^-}{2}\right) &= \frac{dv}{dx}\left(x = \frac{L^+}{2}\right) \\
\Rightarrow\ \frac{-PL^2}{4} &= \frac{-PL^2}{2} + \frac{PL^2}{8} + D_1 \\
\Rightarrow\ D_1 &= \frac{PL^2}{8}
\end{aligned}
$$

Substituting the value of D_1 in Eq.(12), we get

$$
\begin{aligned}
\Rightarrow\ v(x) &= \frac{1}{EI}\left(-\frac{PLx^2}{2} + \frac{Px^3}{6} + \frac{PL^2x}{8} + D_2\right) \\
\text{Use } \mathbf{BC\ 2}\ : v(L) = 0,\ \rightarrow D_2 &= \frac{1}{EI}\left[\frac{PL^3}{2} - \frac{PL^3}{6} - \frac{PL^3}{8}\right] = \frac{5PL^3}{24} \\
\Rightarrow\ v(x) &= \frac{1}{EI}\left(-\frac{PLx^2}{2} + \frac{Px^3}{6} + \frac{PL^2x}{8} + \frac{5PL^3}{24}\right)
\end{aligned}
$$

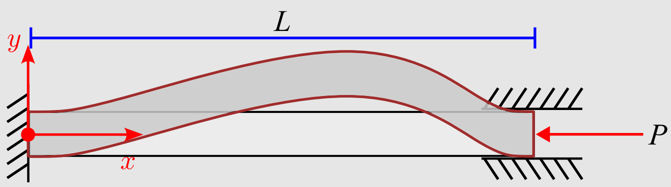
To determine C_2 of Eq.10, we use the continuity of deflection at $x = \frac{L}{2}$

$$
\begin{aligned}
v\left(x = \frac{L}{2}^-\right) &= v\left(x = \frac{L}{2}^+\right) \\
\Rightarrow -\frac{PL^3}{16EI} + C_2 &= \frac{1}{EI}\left(-\frac{PL^3}{8} + \frac{PL^3}{48} + \frac{PL^3}{16} + \frac{5PL^3}{24}\right) \\
\Rightarrow C_2 &= \frac{11PL^3}{48EI}
\end{aligned}
$$

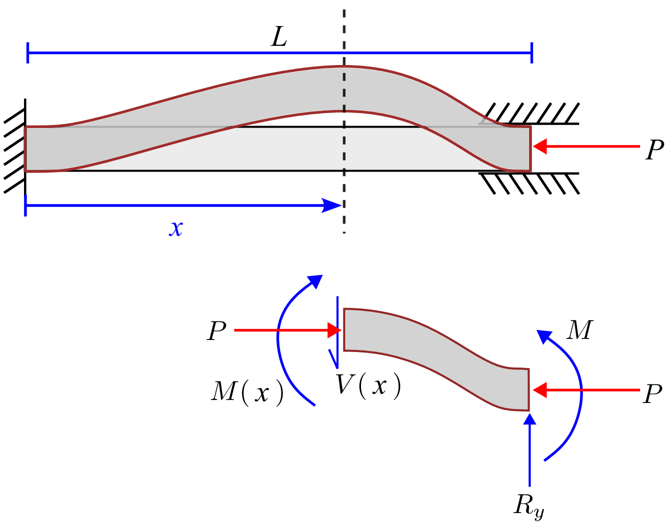
The deflection at any point is therefore

$$
v(x) = \begin{cases} \dfrac{-PLx^2}{4EI} + \dfrac{11PL^3}{48EI} & \text{for } x < \dfrac{L}{2} \\ \dfrac{1}{EI}\left(-\dfrac{PLx^2}{2} + \dfrac{Px^3}{6} + \dfrac{PL^2x}{8} + \dfrac{5PL^3}{24}\right) & \text{for } x \geq \dfrac{L}{2} \end{cases}
$$

Q3. Think of a beam that is clamped against both transverse deflections as well as rotation at both ends (see figure). Deduce the equation which gives us the critical buckling load of the beam. (You don't have to solve it)



Solution:



From force and moment balance

$$M(x) = M - Py + R_y(L - x)$$

The governing equation of EBT is:

$$EI\frac{d^2y}{dx^2} = M(x) = M - Py + R_y(L - x)$$

We have two unknowns in M and R_y at the end and two more unknowns of integrating constants! So, four BCs are required which are:

$$v(0) = v(L) = 0$$
$$\frac{dv}{dx}(0) = \frac{dv}{dx}(L) = 0$$

For the buckling problem, let's first obtain a general solution.

Complimentary function: $v(x) = C_1 \cos\omega x + c_2 \sin\omega x$

Particular integral: $v(x) = \dfrac{M + R_y(L - x)}{P}$

General solution: $v(x) = C_1 \cos\omega x + C_2 \sin\omega x + \dfrac{M + R_y(L - x)}{P}$

i) $v(0) = 0 \;\Rightarrow\; C_1 + \dfrac{M + R_y L}{P} = 0 \;\Rightarrow\; C_1 = -\dfrac{M + R_y L}{P}$

ii) $v(L) = 0 \;\Rightarrow\; C_1 \cos\omega L + C_2 \sin\omega L + \dfrac{M}{P} = 0$

iii) $\dfrac{dv}{dx}(0) = 0 \;\Rightarrow\; C_2\omega - \dfrac{R_y}{P} = 0 \;\Rightarrow\; C_2 = \dfrac{R_y}{P\omega}$

iv) $\dfrac{dv}{dx}(L) = 0 \;\Rightarrow\; -C_1\omega \sin\omega L + C_2\omega \cos\omega L - \dfrac{R_y}{P} = 0$

Using values of C_1 and C_2, relation (ii) becomes:

$$M\,\frac{(1 - \cos\omega L)}{P} + R_y\left[\frac{1}{P\omega}\sin\omega L - \frac{L}{P}\cos\omega L\right] = 0$$

Similarly, relation (iv) becomes:

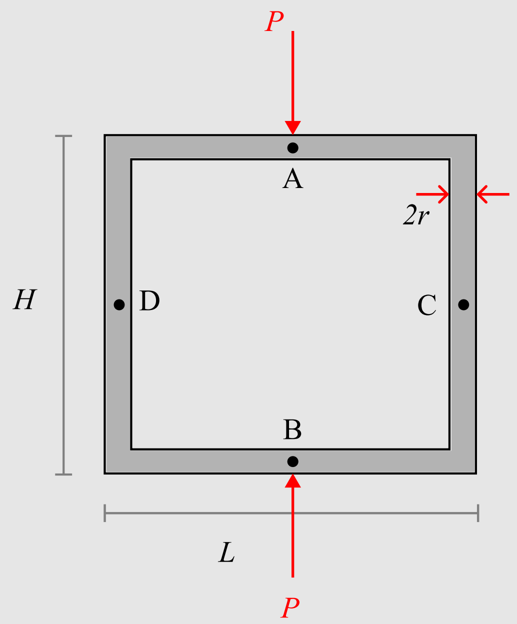
$$M\,\frac{\omega \sin\omega L}{P} + R_y\left[\frac{1}{P}(\cos\omega L - 1) + \frac{L}{P}\omega \sin\omega L\right] = 0$$

Writing them together in a matrix form, we get

$$\begin{bmatrix} \dfrac{1 - \cos\omega L}{P} & \dfrac{1}{P\omega} - \dfrac{L}{P}\cos\omega L \\ \dfrac{\omega \sin\omega L}{P} & \dfrac{(\cos\omega L - 1)}{P} + \dfrac{L}{P}\omega \sin\omega L \end{bmatrix} \begin{bmatrix} M \\ R_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For this to have a non-zero solution, its determinant should be zero. This gives us the equation to obtain the buckling load!

Q4. Think of a rectangular beam as shown below. Assume its cross-section to be solid circular. Suppose the ring is subjected to equal and opposite forces at 'A' and 'B'. Neglect energy in the beam's cross-section due to shear force and axial force.

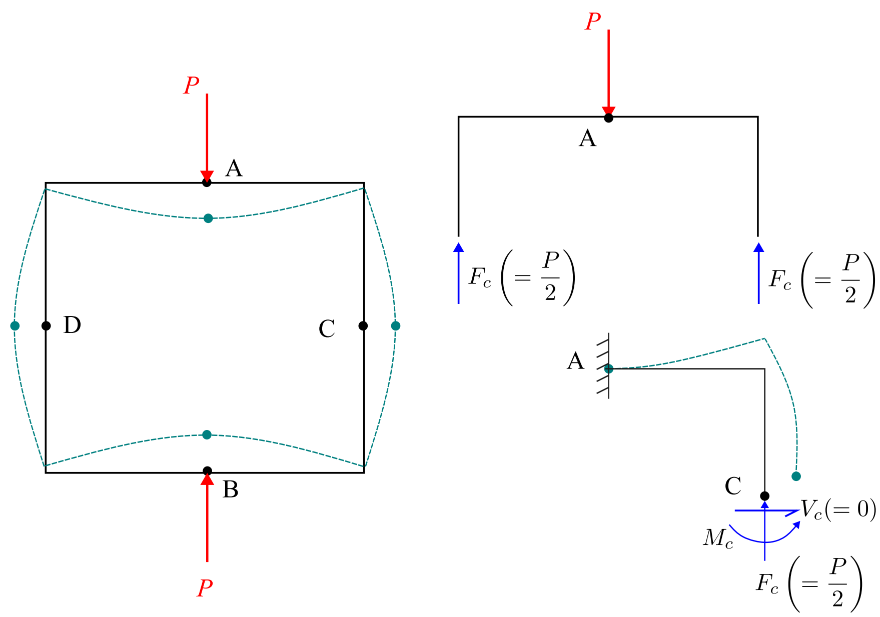


i. By how much will point A and B get closer to each other?

ii. By how much will points C and D get farther apart?

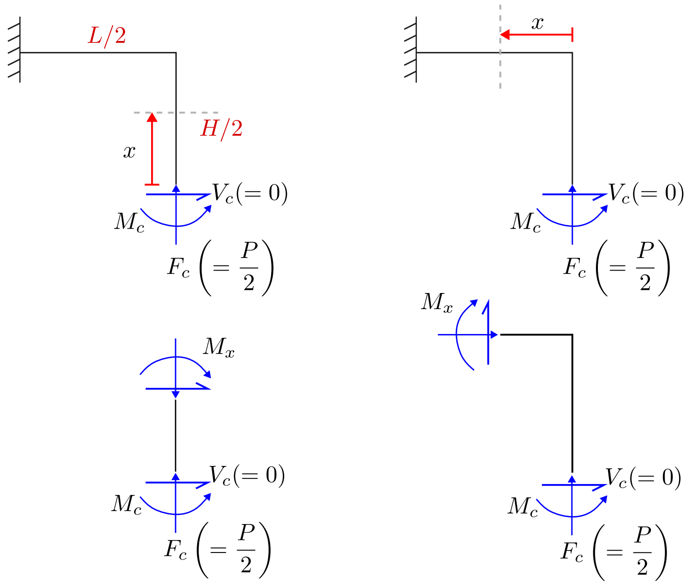
iii. What is the internal moment in the cross-section at C?

Solution:



– Shear force V_c would be zero by symmetry
– $F_c = P/2$ (from force balance in the y-direction)

– No rotation at C implying $\dfrac{\partial U_i}{\partial M_c} = 0$ where M_c is the internal moment at C



Vertical part $\rightarrow M_x = M_c + V_c x$ Horizontal part $\rightarrow M_x = M_c + V_c H/2 + F_c x$

Total strain energy due to bending,

$$U_i(F_c, V_c, M_c) = \int_0^{H/2} \frac{(M_c + V_c x)^2}{2EI} dx + \int_0^{L/2} \frac{(M_c + V_c H/2 + F_c x)^2}{2EI} dx$$

In the above expression, the known variables are $F_c(= P/2)$ and $V_c(= 0)$ while M_c is unknown. Note that M_c is the corresponding moment for rotation at C which we know to be zero. Thus, we can obtain the value of M_c by setting $\dfrac{\partial U_i}{\partial M_c} = 0$ (and then set $V_c = 0,\ F_c = P/2$ after differentiation)

$$\Rightarrow \frac{\partial U_i}{\partial M_c} = 0$$

$$\Rightarrow \int_0^{H/2} \frac{\left(M_c + \cancelto{0}{V_c} x\right)}{EI} dx + \int_0^{L/2} \frac{\left(M_c + \cancelto{0}{V_c} H/2 + \cancelto{P/2}{F_c} x\right)}{EI} dx = 0$$

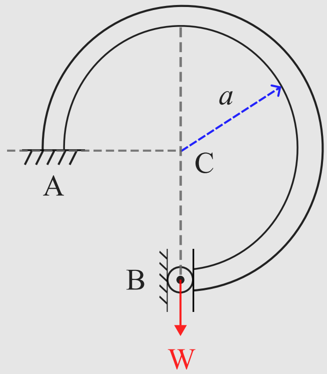
$$\Rightarrow \frac{M_c H}{2EI} + \frac{M_c L}{2EI} + \frac{PL^2}{16EI} = 0$$

$$\Rightarrow M_c = -\frac{PL^2}{8(L+H)}$$

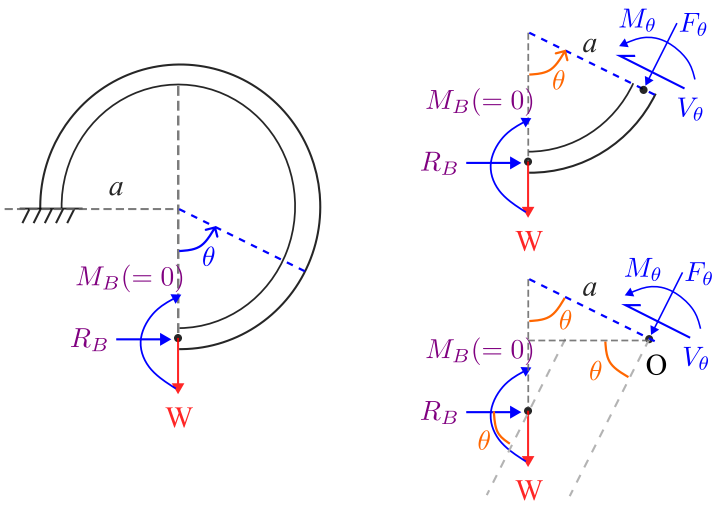
$\delta_{CD} = 2\delta_{Cx}$, where δ_{Cx} is the horizontal displacement of point C.
Get $\delta_{Cx} = \dfrac{\partial U_i}{\partial V_c}$ (and set $V_c = 0,\ F_c = P/2,\ M_c = -\dfrac{PL^2}{8(L+H)}$ after differentiation)

$\delta_{AB} = 2\delta_{Cy}$, where δ_{Cy} is the vertical displacement of point C.
Get $\delta_{Cy} = \dfrac{\partial U_i}{\partial F_c}$ (and set $V_c = 0,\ F_c = P/2,\ M_c = -\dfrac{PL^2}{8(L+H)}$ after differentiation)

Q5. Using energy method, determine (i) the vertical deflection of point B under the action of load W and (ii) the horizontal reaction force at B. The end B is free to rotate but can move only in vertical direction. Consider all forms of energy, i.e. bending, twisting, stretching as well as shearing energy.



Solution:



At location B, there is only horizontal reaction R_B; there is no vertical reaction or moment. Since we are required to find the corresponding rotation at point B, we will introduce a corresponding dummy moment $M_B(=0)$ at B. The knowns are the load W and dummy moment $M_B(=0)$, and the unknown is the reaction R_B. Let's find the moment, shear force, and axial force (M_θ, V_θ, F_θ) at a section at an angle θ: these sectional forces would be used in writing the total stored energy expression later.

Using force balance and moment balance:

$$F_\theta = R_B \cos\theta - W \sin\theta$$
$$V_\theta = R_B \sin\theta + W \cos\theta$$
$$M_\theta = M_B - W a \sin\theta - R_B a(1 - \cos\theta)$$

The total strain energy is

$$
\begin{aligned}
E(F_B, R_B, M_B) &= \int_0^{3\pi/2} \left[\frac{F_\theta^2}{2EA} + \frac{V_\theta^2}{2kGA} + \frac{M_\theta^2}{2EI} \right] a d\theta \\
&= \int_0^{3\pi/2} \left[\frac{(R_B \cos\theta - W \sin\theta)^2}{2EA} + \frac{(R_B \sin\theta + W \cos\theta)^2}{2kGA} \right. \\
&\qquad \left. + \frac{(M_B - W a \sin\theta - R_B a (1-\cos\theta))^2}{2EI} \right] a d\theta
\end{aligned}
$$

Since the beam has no horizontal deflection at B, we can determine the corresponding force R_B by setting $\frac{\partial E}{\partial R_B} = 0$ and evaluating at $M_B = 0$ and given W.

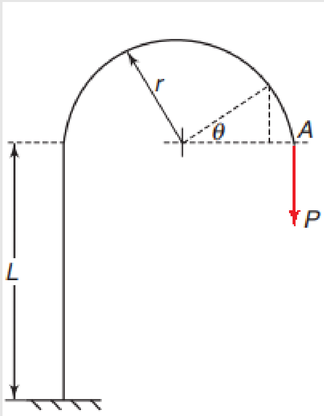
$$
\begin{aligned}
&\Rightarrow \frac{\partial U_i}{\partial R_B} = 0 \\
&\Rightarrow \int_0^{3\pi/2} \left[\frac{F_\theta}{EA}\frac{\partial F_\theta}{\partial R_B} + \frac{V_\theta}{kGA}\frac{\partial V_\theta}{\partial R_B} + \frac{M_\theta}{EI}\frac{\partial M_\theta}{\partial R_B} \right] a d\theta = 0 \\
&\Rightarrow \int_0^{3\pi/2} \frac{R_B \cos\theta - W \sin\theta}{EA} \cos\theta + \frac{R_B \sin\theta + W\cos\theta}{kGA} \sin\theta \\
&\qquad + \frac{\cancelto{0}{M_B} - W a \sin\theta - R_B a (1-\cos\theta)}{EI} (-a(1-\cos\theta))\, a d\theta = 0 \\
&\Rightarrow a \left[\frac{3\pi R_B - 2W}{4EA} + \frac{3\pi R_B + 2W}{4kGA} + \frac{W a^2}{2EI} + R_B a^2 \left(2 + \frac{9\pi}{4}\right) \right] = 0 \\
&\Rightarrow R_B \left[\frac{3\pi}{4EA} + \frac{3\pi}{4kGA} + a^2\left(2 + \frac{9\pi}{4}\right) \right] = W \left[\frac{1}{2EA} - \frac{1}{2kGA} - \frac{a^2}{2EI} \right] \\
&\Rightarrow R_B = W \frac{\left[\frac{1}{2EA} - \frac{1}{2kGA} - \frac{a^2}{2EI} \right]}{\left[\frac{3\pi}{4EA} + \frac{3\pi}{4kGA} + a^2\left(2 + \frac{9\pi}{4}\right) \right]}
\end{aligned}
$$

To obtain vertical deflection at B, differentiate the energy with respect to the corresponding force W

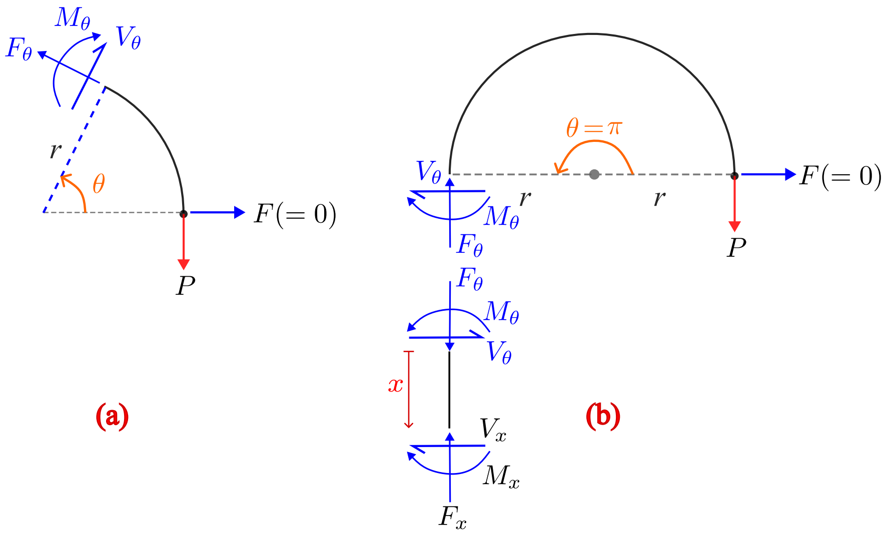
$$
\begin{aligned}
\Rightarrow\ \delta_B &= \frac{\partial U_i}{\partial W} \\
&= \int_0^{3\pi/2} \left[\frac{F_\theta}{EA}\frac{\partial F_\theta}{\partial W} + \frac{V_\theta}{kGA}\frac{\partial V_\theta}{\partial W} + \frac{M_\theta}{EI}\frac{\partial M_\theta}{\partial w} \right] a d\theta \\
&= \int_0^{3\pi/2} \frac{R_B \cos\theta - W \sin\theta}{EA} (-\sin\theta) + \frac{R_B \sin\theta + W\cos\theta}{kGA} \cos\theta \\
&\qquad + \frac{\cancelto{0}{M_B} - W a \sin\theta - R_B a (1-\cos\theta)}{EI} (a \sin\theta))\, a d\theta
\end{aligned}
$$

Put the value of R_B as obtained previously and solve this in a similar way to obtain δ_B!

Q6. For the structure shown, what is the vertical deflection at end A? Also, determine the ratio of L to r if the horizontal and vertical deflections of the loaded end A are equal. P is the only force acting.



Solution: We are required to find both vertical and horizontal deflection of the beam at point A. We are given the corresponding force P at A to obtain the corresponding vertical deflection. However, there is no corresponding force for the horizontal displacement. Hence, we apply a dummy horizontal force $F(=0)$.



Next, we find the sectional forces and moments at an arbitrary cross-section. We take two cross-sections, one at an angle θ from A (see subfigure (a)) and another at a distance x (see subfigure (b)).

The procedure is as follows:
– First find out the internal shear force, axial force, bending moment, and twisting moment.
– Write down energy
– Take derivative with respect to the corresponding force to find the corresponding displacement.

In this problem, we neglect the energy due to shearing and stretching and only focus on energy due to bending. So we will only use the sectional bending moment for calculating the energy.

Bending moment at any section on the curved semi-circle:

$$
\begin{aligned}
&\circlearrowleft\!\!\!+ \sum M_o = 0\\
&\Rightarrow\ -M_\theta - Pr\left(1-\cos\theta\right) + Fr\sin\theta = 0\\
&\Rightarrow\ M_\theta = -Pr\left(1-\cos\theta\right) + Fr\sin\theta
\end{aligned}
$$

Bending moment at any section on the straight portion:

$$
\begin{aligned}
&\circlearrowleft\!\!\!+ \sum M_o = 0\\
&\Rightarrow\ M_x - M_\theta\mid_{\theta=\pi} + V_\theta\mid_{\theta=\pi} x = 0\\
&\Rightarrow\ M_x = -2Pr - Fx
\end{aligned}
$$

$$
\begin{aligned}
U_i\left(P,F\right) &= \int_0^\pi \frac{M_\theta^2}{2EI} r d\theta + \int_0^L \frac{M_x^2}{2EI} dx\\
&= \int_0^\pi \frac{\left(-Pr\left(1-\cos\theta\right) + Fr\sin\theta\right)^2}{2EI} r d\theta + \int_0^L \frac{\left(-2Pr - Fr\right)^2}{2EI} dx
\end{aligned}
$$

To obtain vertical deflection, δ_v, at point A, we take partial derivative of the energy w.r.t to the corresponding force P:

$$
\begin{aligned}
\frac{\partial U_i}{\partial P}\Big|_{F=0} &= \int_0^\pi \frac{\left(-Pr\left(1-\cos\theta\right) + \cancelto{0}{F}x\sin\theta\right)}{EI}\left(-r(1-\cos\theta)\right) r d\theta + \int_0^L \frac{-2Pr - \cancelto{0}{F}r}{EI}(-2r)dx\\
&= \frac{Pr^3}{EI}\int_0^\pi \left(1-\cos\theta\right)^2 d\theta + \frac{4Pr^2L}{EI}\\
&= \frac{Pr^3}{EI}\left(\frac{3\theta}{2} - 2\sin\theta + \frac{1}{4}\sin 2\theta\right)\Big|_0^\pi + \frac{4Pr^2L}{EI}\\
&= \frac{3\pi Pr^3}{2EI} + \frac{4Pr^2L}{EI}\\
&= \frac{Pr^2}{EI}\left[\frac{3\pi r}{2} + 4L\right]
\end{aligned}
$$

To obtain horizontal deflection, δ_h, at point A, we take the partial derivative of the energy w.r.t to the corresponding dummy force F

$$
\begin{aligned}
\frac{\partial U_i}{\partial F}\Big|_{F=0} &= \int_0^\pi \frac{\left(-Pr\left(1-\cos\theta\right) + \cancelto{0}{F}r\sin\theta\right)}{EI}\left(r\sin\theta\right) r d\theta + \int_0^L \frac{-2Pr - \cancelto{0}{F}x}{EI}(-x)dx\\
&= -\frac{Pr^3}{EI}\int_0^\pi \left(1-\cos\theta\right)\sin\theta d\theta + \frac{2Pr}{EI}\int_0^L x dx\\
&= -\frac{Pr^3}{EI}\left(\frac{\cos^2\theta}{2} - \cos\theta\right)\Big|_0^\pi + \frac{2Pr}{EI}\left(\frac{x^2}{2}\right)\Big|_0^L
\end{aligned}
$$

$$= -\frac{2Pr^3}{EI} + \frac{2PrL^2}{EI}$$
$$= \frac{Pr}{EI}\left[-2r^2 + 2L^2\right]$$

Equating δ_v to δ_h, we get

$$\frac{Pr^2}{EI}\left[\frac{3\pi r}{2} + 4L\right] = \frac{Pr}{EI}\left[-2r^2 + 2L^2\right]$$
$$L^2 - 4Lr - r^2\left(\frac{3\pi}{2} + 2\right) = 0$$

Dividing by r^2 and putting $\frac{L}{r} = \rho$, we get

$$\rho^2 - 4\rho - \left(\frac{3\pi}{2} + 2\right) = 0$$

Upon solving the above, we get

$$\rho = \frac{4 \pm \sqrt{16 + 4(3\pi/2 + 2)}}{2} = 2 + \sqrt{6 + \frac{3}{2}\pi}.$$