

Stress Invariants

Recall the cubic equation we obtained for the stress eigenvalue problem

$$[\underline{\sigma}][\underline{n}] = \lambda [\underline{n}]$$

$$\Rightarrow \det([\underline{\sigma}] - \lambda [\underline{I}]) = 0$$

$$\Rightarrow -\lambda^3 + I_1 \lambda^2 - I_2 \lambda + I_3 = 0$$

STRESS INVARIANTS

$$\left\{ \begin{array}{l} I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33} \\ I_2 = \begin{vmatrix} \sigma_{11} & \tau_{12} \\ \tau_{12} & \sigma_{22} \end{vmatrix} + \begin{vmatrix} \sigma_{22} & \tau_{23} \\ \tau_{23} & \sigma_{33} \end{vmatrix} + \begin{vmatrix} \sigma_{11} & \tau_{13} \\ \tau_{13} & \sigma_{33} \end{vmatrix} \\ I_3 = \begin{vmatrix} \sigma_{11} & \tau_{12} & \tau_{13} \\ \tau_{12} & \sigma_{22} & \tau_{23} \\ \tau_{13} & \tau_{23} & \sigma_{33} \end{vmatrix} \end{array} \right.$$

$$\left. \begin{array}{l} I_1 \rightarrow \text{1st stress invariant} \\ I_2 \rightarrow \text{2nd stress invariant} \\ I_3 \rightarrow \text{3rd stress invariant} \end{array} \right\} \begin{array}{l} \text{These values } \textcolor{red}{\text{do not}} \\ \textcolor{red}{\text{change}} \text{ with the} \\ \text{choice of coordinate} \\ \text{system} \end{array}$$

In other words, if you choose a different coordinate system $\hat{e}_1 - \hat{e}_2 - \hat{e}_3$, where the stress components are say $\hat{\sigma}_{11}, \hat{\sigma}_{22}, \hat{\sigma}_{33}, \hat{\tau}_{12}, \hat{\tau}_{13}, \hat{\tau}_{23}$, still the invariants will remain the same.

$$\text{e.g. } \hat{I}_1 = \hat{\sigma}_{11} + \hat{\sigma}_{22} + \hat{\sigma}_{33} = I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33}$$

Using principal stresses

$$I_1 = \lambda_1 + \lambda_2 + \lambda_3$$

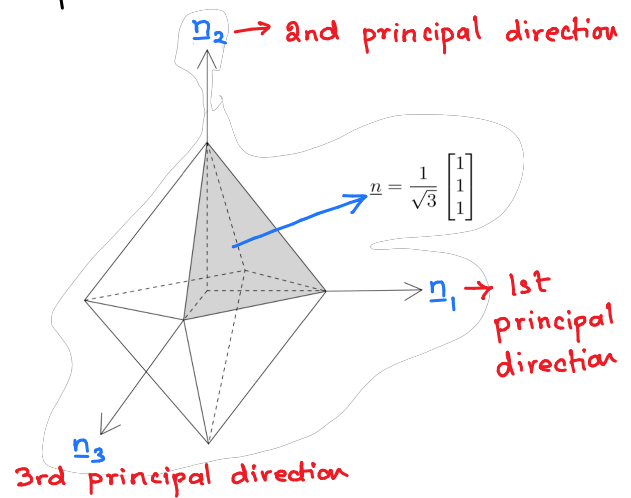
$$I_2 = \lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_1 \lambda_3$$

$$I_3 = \lambda_1 \lambda_2 \lambda_3$$

Octahedral Stresses

First we need to know what an octahedral plane is : they are faces of an octahedron having 8 faces whose normals have the form

$$[\underline{n}] = \begin{pmatrix} \underline{n}_1 \\ \underline{n}_2 \\ \underline{n}_3 \end{pmatrix} = \begin{bmatrix} \pm \frac{1}{\sqrt{3}} \\ \pm \frac{1}{\sqrt{3}} \\ \pm \frac{1}{\sqrt{3}} \end{bmatrix}$$



in the coordinate system of principal directions i.e. they are equally inclined from all principal directions.

The normal and shear components of traction on the octahedral faces are called **octahedral stress components**. The normal component of traction on octahedral planes will be

$$\begin{aligned} \sigma_{oct} &= ([\underline{\sigma}][\underline{n}]) \cdot [\underline{n}] \\ &= \left(\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} \pm \frac{1}{\sqrt{3}} \\ \pm \frac{1}{\sqrt{3}} \\ \pm \frac{1}{\sqrt{3}} \end{bmatrix} \right) \cdot \begin{bmatrix} \pm \frac{1}{\sqrt{3}} \\ \pm \frac{1}{\sqrt{3}} \\ \pm \frac{1}{\sqrt{3}} \end{bmatrix} \\ &= \frac{\lambda_1 + \lambda_2 + \lambda_3}{3} = \frac{I_1}{3} \leftarrow \text{invariant} \end{aligned}$$

For stress matrix expressed in a general coordinate system,

$$[\underline{\sigma}] = \begin{bmatrix} \sigma_{11} & \tau_{12} & \tau_{13} \\ \text{sym} & \sigma_{22} & \tau_{23} \\ & & \sigma_{33} \end{bmatrix}, \quad \sigma_{oct} = \frac{I_1}{3} = \frac{\sigma_{11} + \sigma_{22} + \sigma_{33}}{3}$$

We can also find the shear stress component on octahedral planes

$$\begin{aligned}\tau_{oct}^2 &= \|\underline{\sigma} \underline{n}\|^2 - \sigma_{oct}^2 \\&= \left\| \begin{bmatrix} \pm \lambda_1 / \sqrt{3} \\ \pm \lambda_2 / \sqrt{3} \\ \pm \lambda_3 / \sqrt{3} \end{bmatrix} \right\|^2 - \left(\frac{\lambda_1 + \lambda_2 + \lambda_3}{3} \right)^2 \\&= \frac{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}{3} - \frac{(\lambda_1 + \lambda_2 + \lambda_3)^2}{9} \\&= \frac{2(\lambda_1 + \lambda_2 + \lambda_3)^2 - 6(\lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1)}{9} \\&= \frac{2I_1^2 - 6I_2}{9}\end{aligned}$$

$$\Rightarrow \tau_{oct} = \frac{\sqrt{2}}{3} \sqrt{I_1^2 - 3I_2} \quad \left. \vphantom{\tau_{oct}} \right\} \begin{array}{l} \text{turns out invariant} \\ \text{as well since } I_1^2 \\ \text{and } I_2 \text{ are invariants} \end{array}$$

Key points : a) Octahedral planes are defined w.r.t principal directions and not with any arbitrary frame of reference i.e.

$$[\underline{n}] \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{bmatrix} \pm 1/\sqrt{3} \\ \pm 1/\sqrt{3} \\ \pm 1/\sqrt{3} \end{bmatrix}$$

b) We have the same σ_{oct} and τ_{oct} on all 8 faces of the octahedron

c) If the first invariant $I_1 = 0$, then normal stresses on the octahedral planes are zero and only shear acts

State of pure shear

The state of stress at a point can be characterized by 6 independent stress components. The magnitudes of these components depend upon the choice of the coordinate sys.

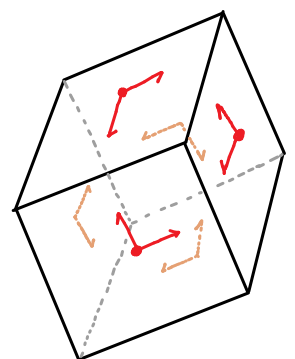
If, for at least one particular choice of coordinate system, we find that $\sigma_{11} = \sigma_{22} = \sigma_{33} = 0$, then a state of pure shear is said to exist at that point. In that particular coordinate system, the stress matrix will be

$$[\underline{\underline{\sigma}}]_{\begin{pmatrix} \underline{e}_1 \\ \underline{e}_2 \\ \underline{e}_3 \end{pmatrix}} = \begin{bmatrix} 0 & \tau_{12} & \tau_{13} \\ \tau_{12} & 0 & \tau_{23} \\ \tau_{13} & \tau_{23} & 0 \end{bmatrix}$$

For this coordinate system, $I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33} = 0$.

But since I_1 is an invariant and does not depend upon the choice of coordinate system, $I_1 = 0$ for any choice of coordinate system. Hence, the condition needed for state of pure shear to exist at a point is to have $I_1 = 0$

State of
pure shear
in some coordinate
system



Hydrostatic and Deviatoric (pure shear) parts of stress tensor

An arbitrary state of stress can be decomposed into a hydrostatic state of stress and a state of pure shear.
deviatoric

$$\underline{\underline{\sigma}} = \underbrace{\frac{1}{3} I_1(\underline{\underline{\sigma}}) \underline{\underline{I}}}_{\underline{\underline{\sigma}}_{\text{hyd}}} + \underbrace{\left(\underline{\underline{\sigma}} - \frac{1}{3} I_1(\underline{\underline{\sigma}}) \underline{\underline{I}} \right)}_{\underline{\underline{\sigma}}_{\text{dev}}}$$

The hydrostatic stress tensor $\frac{1}{3} I_1(\underline{\underline{\sigma}}) \underline{\underline{I}}$ is proportional to the identity tensor $\underline{\underline{I}}$. The reason it is called hydrostatic is because it has same normal stress acting in all three faces, just like pressure acting at a point in a static fluid.

$$\begin{bmatrix} \sigma_{11} & \tau_{12} & \tau_{13} \\ \tau_{12} & \sigma_{22} & \tau_{23} \\ \tau_{13} & \tau_{23} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} \overset{p = I_1/3}{\textcircled{p}} & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix} + \begin{bmatrix} \sigma_{11} - p & \tau_{12} & \tau_{13} \\ \tau_{12} & \sigma_{22} - p & \tau_{23} \\ \tau_{13} & \tau_{23} & \sigma_{33} - p \end{bmatrix}$$

The deviatoric part has first stress invariant as zero

$$\begin{aligned} I_1(\underline{\underline{\sigma}}_{\text{dev}}) &= (\sigma_{11} - p) + (\sigma_{22} - p) + (\sigma_{33} - p) \\ &= \sigma_{11} + \sigma_{22} + \sigma_{33} - 3p \\ &= \sigma_{11} + \sigma_{22} + \sigma_{33} - 3 \left(\frac{\sigma_{11} + \sigma_{22} + \sigma_{33}}{3} \right) = 0 \end{aligned}$$

The hydrostatic stress tries to change the size of cuboid without distorting the shape while the deviatoric stress tries to distort the shape of cuboid without changing the size/volume of cuboid

