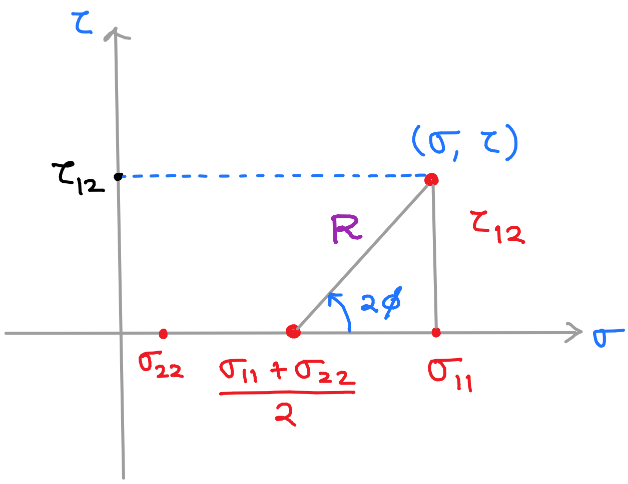
Mohr's Circle

$$\sigma_n = \frac{\sigma_{11} + \sigma_{22}}{2} + R\cos(2\phi - 2\theta)$$

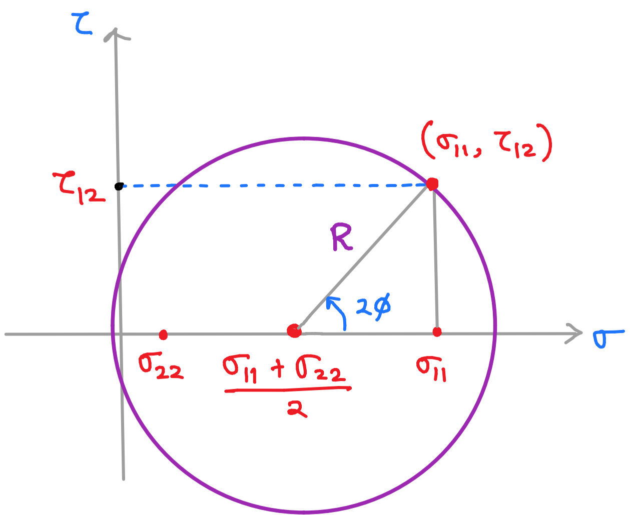
$$\tau_n = R\sin(2\phi - 2\theta)$$

Based on these formulas, lets try to obtain the locus of σ_n and τ_n for all values of θ.

Let us think of a $\sigma - \tau$ plane and plot σ_n and τ_n for each value of θ in this plane. The plane has σ on x-axis and τ on the y-axis.

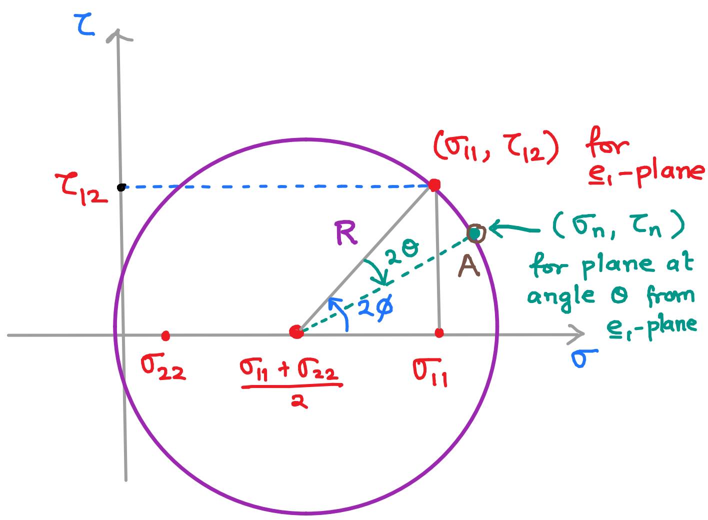


From the above relations, one can figure out that the centre of the circle on the σ-axis will be at $\left(\frac{\sigma_{11} + \sigma_{22}}{2}, 0\right)$. We can place σ_{11} and σ_{22} on the σ-axis and τ_{12} on the τ-axis. Then, we plot the point (σ_{11}, τ_{12}) which corresponds to $\underline{e}_1$-plane. If you join this point with the center, the line obtained will give us the radius of the circle, which turns out to be R



Once we obtain the radius and center of the circle, we can draw the complete circle $\longrightarrow$ is called the Mohr's circle

The 2D Mohr's circle can be used to find the 2D state of stress on any plane



$$\sigma_n = \frac{\sigma_{11} + \sigma_{22}}{2} + R\cos(2\phi - 2\theta)$$

$$\tau_n = R\sin(2\phi - 2\theta)$$

While we can use the above equations for finding σ and τ on any arbitrary plane at an angle θ. For using the Mohr's circle, we note the angle in the cosine and sine terms is $2\phi - 2\theta$. So the radial line from the center to the point corresponding to the θ-plane on the Mohr's circle should be at an angle of $(2\phi - 2\theta)$ from the $\underline{e}_1$-plane. In other words, we can obtain the point corresponding to the θ-plane on Mohr's circle by going in the clockwise direction by angle 2θ from the $\underline{e}_1$-plane point. So the radial line from the center to the point A corresponds to the θ-plane with point A stresses σ_{11}', τ_{12}'.

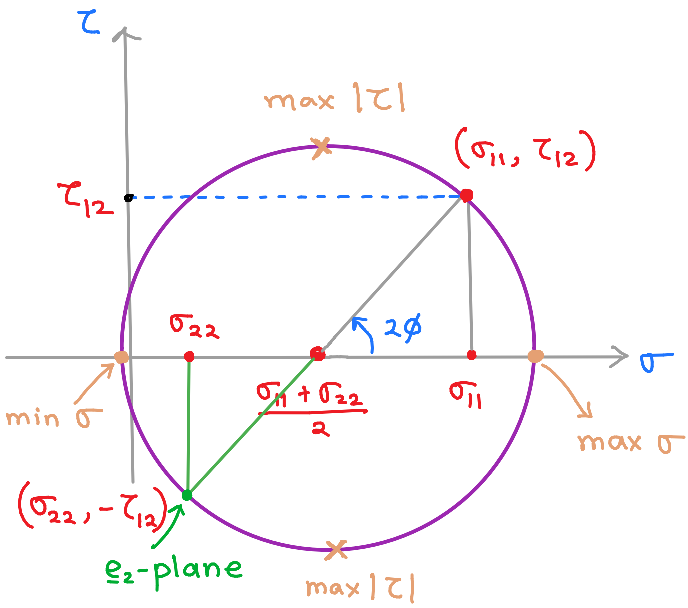
Steps for drawing Mohr's circle and for finding the point corresponding to θ-plane

1. Draw the center of the circle at $\left(\frac{\sigma_{11} + \sigma_{22}}{2}, 0\right)$
2. Draw (σ, τ) for $\underline{e}_1$-plane, i.e., the point (σ_{11}, τ_{12})
3. Draw a line joining the center and the point (σ_{11}, τ_{12}) to get the radius of the circle
4. With the center and radius known, draw the Mohr's circle
5. To find (σ, τ) for θ-plane, rotate the radial line of $\underline{e}_1$-plane by 2θ clockwise

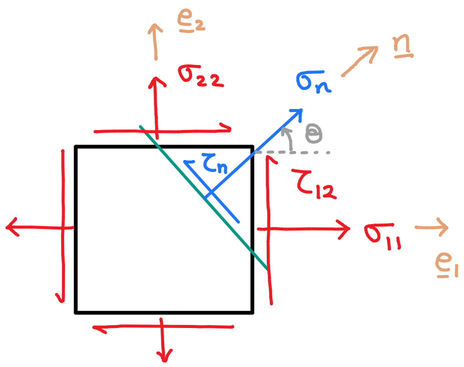
Notice that the normal to the θ-plane is at angle θ with $\underline{e}_1$ in counterclockwise direction. But on the Mohr's circle, we draw that point by rotating 2θ in clockwise direction from the point corresponding to $\underline{e}_1$-plane. This is because we have 2θ with a minus sign in the argument of trigonometric functions

Sign Convention while using Mohr's circle

Once we have determined the point on Mohr's circle corresponding to $\underline{e}_1$-plane, we can obtain the $(\sigma-\tau)$ corresponding to the $\underline{e}_2$-plane by rotating $2\times 90°$ in the clockwise direction from $\underline{e}_1$-plane. Thus we get the $\underline{e}_2$-plane at the diametrically opposite point w.r.t. the $\underline{e}_1$-plane.



The point for $\underline{e}_2$-plane has coordinates $(\sigma_{22}, -\tau_{12})$. However, we know the shear stress on the $\underline{e}_2$-plane is τ_{12}. So why are we getting $-\tau_{12}$ from the Mohr's circle? This is because of our convention for the sign of τ_n is taken as +ve if τ_n is acting $90°$ ccw direction from its plane normal $\underline{n}$.

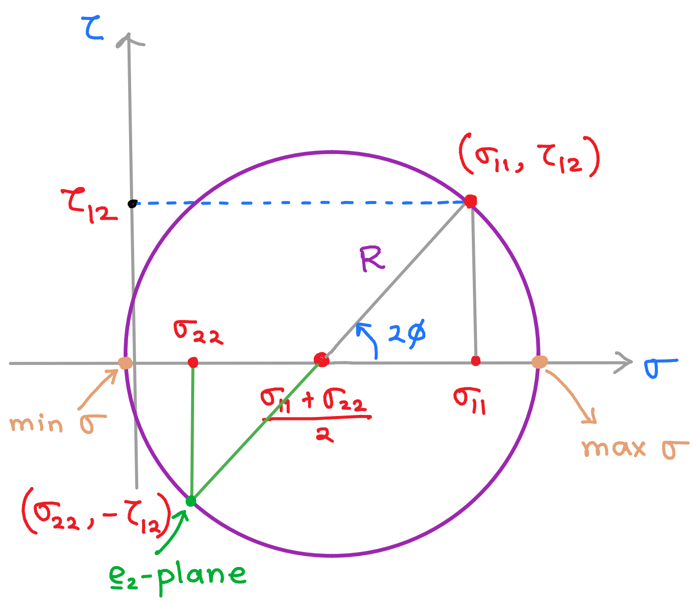


If we go $90°$ in the CCW direction from the plane normal $\underline{e}_1$ τ_n would be pointing towards $-\underline{e}_1$ on the $\underline{e}_2$-plane. So, Mohr's circle is giving us τ on $\underline{e}_2$-plane along $-\underline{e}_1$ direction, whereas τ_{12}, by definition, is the shear component along $+\underline{e}_1$-direction. Therefore, Mohr's circle gives us $-\tau_{12}$ for shear traction on $\underline{e}_2$-plane.

Other conclusions that can be drawn using Mohr's circle

We can get the maximum and minimum values of σ and τ

- The maximum/minimum values of σ are plotted on σ-axis itself. The maximum value of σ will correspond to the principal stress λ_1 and the min value of σ to λ_2



$$\lambda_1 = \frac{\sigma_{11}+\sigma_{22}}{2} + R \quad , \quad \lambda_2 = \frac{\sigma_{11}+\sigma_{22}}{2} - R$$

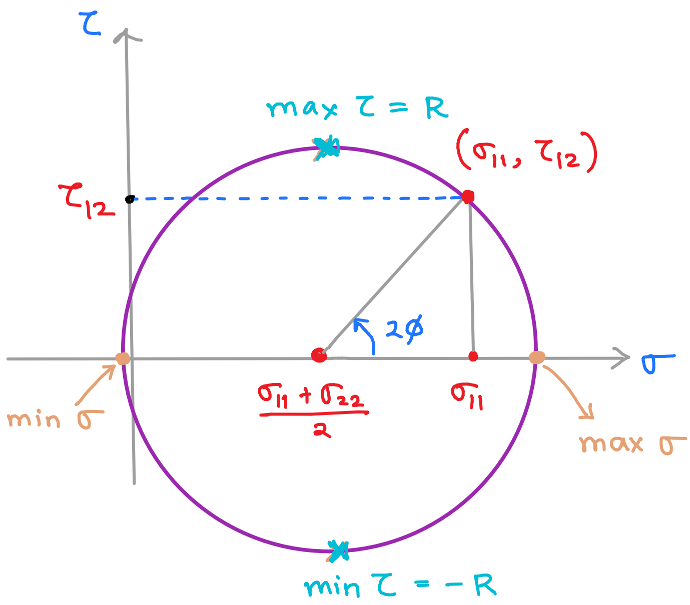
This allows us to get the values of principal stress components directly from the Mohr's circle. One can also write the center and radius of the circle in terms of principal stresses λ_1 and λ_2

$$\text{Center} = \left(\frac{\lambda_1+\lambda_2}{2}, 0\right) \quad , \quad \text{Radius} = \frac{\lambda_1-\lambda_2}{2}$$

- The max/min values of shear are equal to the radius of the Mohr's circle and occur on the τ-axis

$$\tau_{max} = R = \frac{\lambda_1 - \lambda_2}{2} \quad \text{(top)}$$

$$\tau_{min} = -R = -\frac{\lambda_1 - \lambda_2}{2} \quad \text{(bottom)}$$



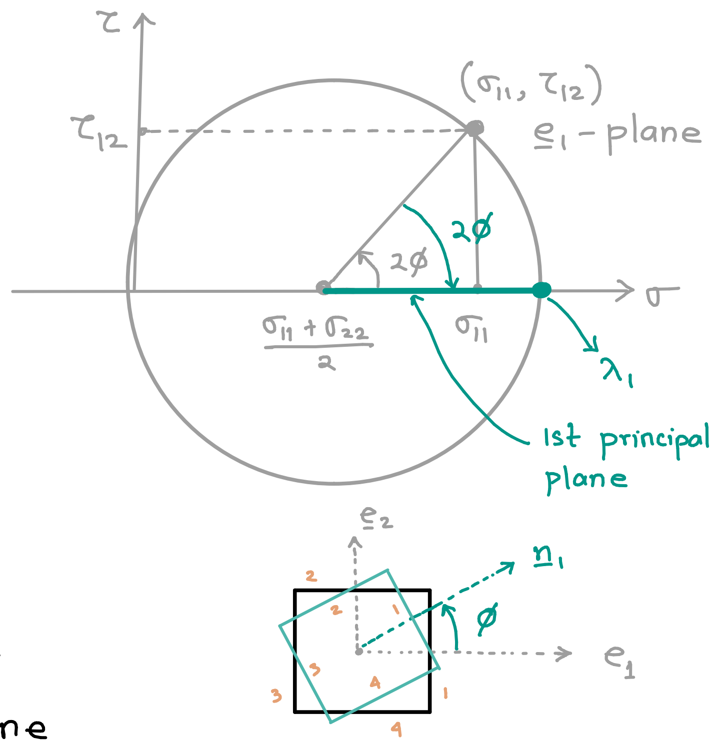
Also, note that the normal stress on the planes having maximum shear stress (topmost and bottommost points of the circle) was the σ corresponding to the center of the circle i.e. $\frac{\sigma_{11}+\sigma_{22}}{2}$ or $\frac{\lambda_1+\lambda_2}{2}$

So the coordinates of the topmost pt of the circle is $\left(\frac{\lambda_1+\lambda_2}{2}, \frac{\lambda_1-\lambda_2}{2}\right)$

How do you find the planes of principal stresses from Mohr's circle?

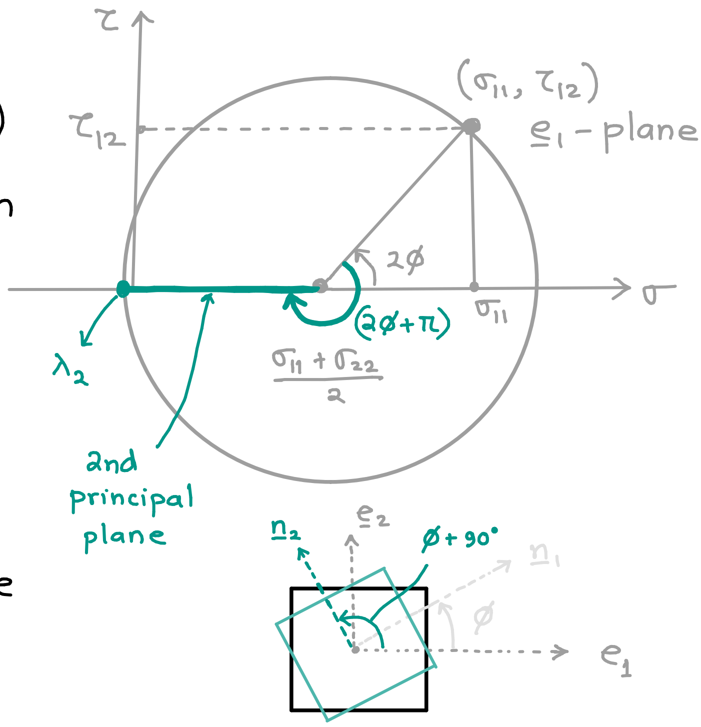
- We need to go 2ϕ clockwise from the e_1-plane to reach the principal plane, where λ_1 is acting, on the Mohr's circle

- That means in the original coordinate system, we need to go by an angle of ϕ in the CCW direction from the e_1-plane to get to the first principal plane



- Similarly, for the 2nd principal plane, we have to go $(2\phi + 180°)$ clockwise in the Mohr's circle from e_1-plane.

- In actual coordinate system, we need to go by an angle of $(\phi + 90°)$ in the CCW direction from e_1-plane

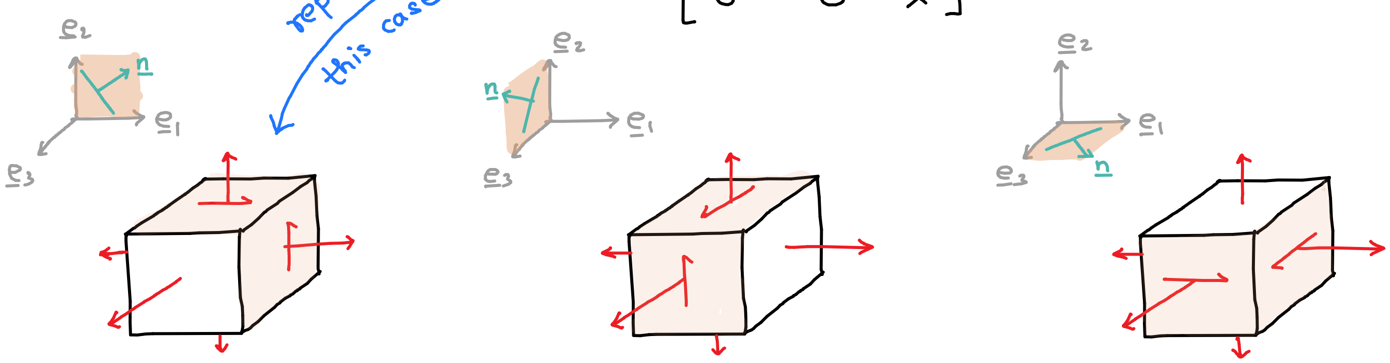


<u>Limitation of 2D Mohr's circle</u>

Mohr's circle is only applicable to finding normal and shear components on planes that are perpendicular to one of the principal directions. As such, we must consider a coordinate system s.t. the third coordinate axis is along one of the principal directions. The stress matrix in this coordinate system will have zero shear components in the third row and column

$$[\underline{\sigma}] = \begin{bmatrix} \times & \times & 0 \\ \times & \times & 0 \\ 0 & 0 & \times \end{bmatrix}$$

represents this case



$\Downarrow$ You can draw 2D Mohr's circle for e_1-e_2 plane $\underline{n \perp e_3}$

$\Downarrow$ You can draw 2D Mohr's circle for e_2-e_3 plane $\underline{n \perp e_1}$

$\Downarrow$ You can draw 2D Mohr's circle for e_1-e_3 plane $\underline{n \perp e_2}$

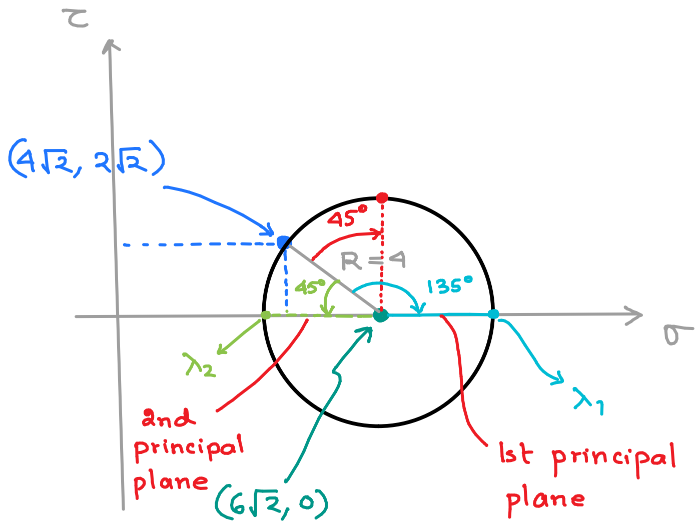
Example : Consider the following stress matrix

$$[\underline{\sigma}] = \begin{bmatrix} 4\sqrt{2} & 2\sqrt{2} & 0 \\ 2\sqrt{2} & 8\sqrt{2} & 0 \\ 0 & 0 & 10 \end{bmatrix}$$

- Find principal planes, planes of maximum shear stress, and also their values

Soln : We can see from the stress matrix that the $\underline{e}_3$-axis is aligned with the principal axis and hence we can use 2D Mohr's circle for the plane spanned by $\underline{e}_1 - \underline{e}_2$.

- First plot (σ_{11}, τ_{12}) on σ-τ plot
- Next plot the center
$$\left(\frac{\sigma_{11}+\sigma_{22}}{2}, 0\right) = (6\sqrt{2}, 0)$$
- Radius $R = \sqrt{(2\sqrt{2})^2 + (2\sqrt{2})^2} = 4$



- After drawing the Mohr's circle, the principal stress components and τ_{max} are:

$$\lambda_{1,2} = (\sigma \text{ at center}) \pm R = 6\sqrt{2} \pm 4$$

$$\tau_{max} = R = 4$$

Plane direction from $\underline{e}_1$-plane	Mohr's circle	In actual coordinates
– 1st principal plane	135° CW	$\frac{135°}{2}$ CCW
– 2nd principal plane	45° CCW	22.5° CW
– Max shear plane	45° CW	22.5 CCW

3D Mohr's Circle and Stress planes

Suppose that we express the stress tensor using the principal directions. Then the stress matrix will be diagonal

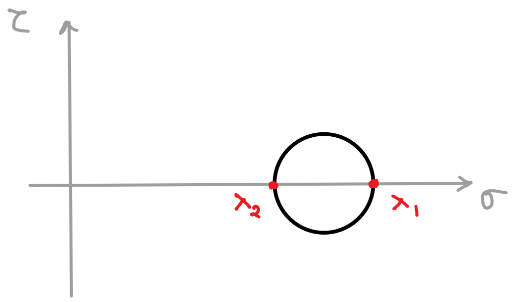
$$[\underline{\sigma}]_{(\underline{n}_1,\underline{n}_2,\underline{n}_3)} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

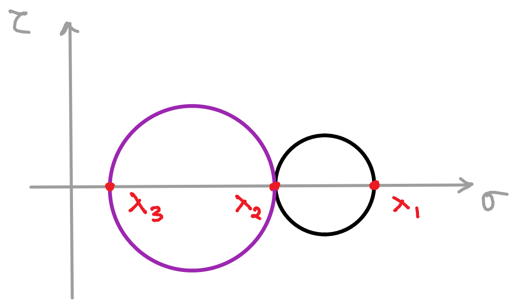
principal csys

Assume $\lambda_1 > \lambda_2 > \lambda_3$

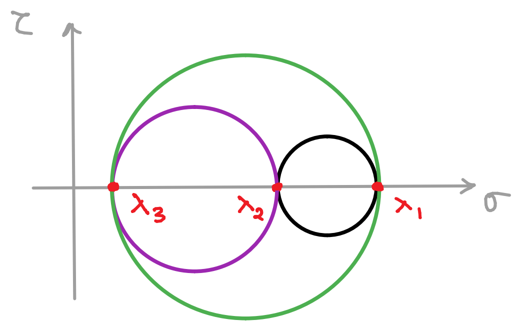
With this stress matrix, we will now think of arbitrary planes and plot $\sigma - \tau$ on those planes. We will NOT confine ourselves to planes whose normal is perpendicular to one of the principal axes.

However, let us first consider the planes whose normal is $\perp$ to $\underline{n}_3$ axis. If we start plotting $(\sigma - \tau)$ on such planes, we will get the Mohr's circle passing through λ_1 and λ_2

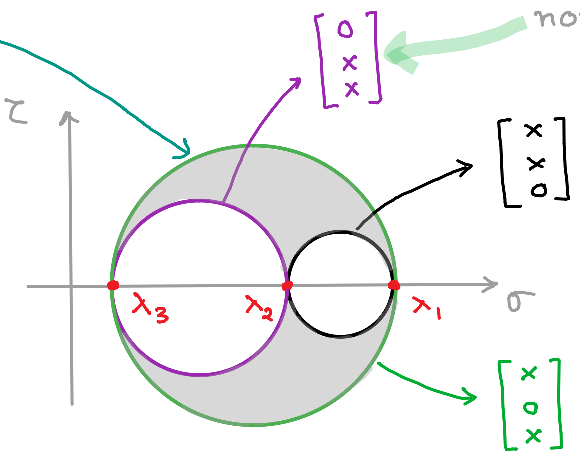




Similarly, we can draw the Mohr's circle corresponding to planes whose normals are perpendicular to 1st principal axis, $\underline{n}_1$: this circle passes through λ_2 and λ_3. Then we take the planes with normals perpendicular to the 2nd principal direction and we get Mohr's circle through λ_1 and λ_3



This will be the biggest circle of the three circles ($\because$ $\lambda_1 \geq \lambda_2 \geq \lambda_3$)



normal vector for a (σ, τ) plane for $\lambda_2 - \lambda_3$ Mohr's circle expres in n_1-n_2-n_3 csys

These three circles correspond to very specific normal directions i.e., they have one of their components zero. If we now plot (σ, τ) for all planes with arbitrary normal directions $\begin{bmatrix} \times \\ \times \\ \times \end{bmatrix}$, we will get the shaded region. This σ-τ plot is called the Mohr's stress planes or 3D Mohr's circle

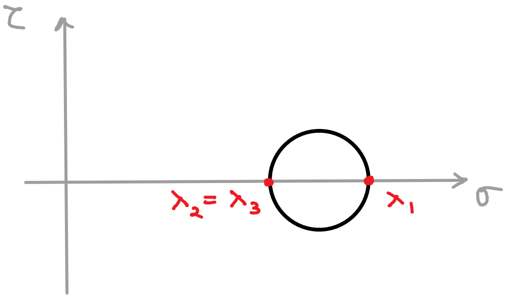
Absolute Maximum Shear Stress can be obtained from the 3D Mohr's circle:

$$\tau_{max} = \frac{\lambda_1 - \lambda_3}{2} \qquad (\because \lambda_1 > \lambda_2 > \lambda_3)$$

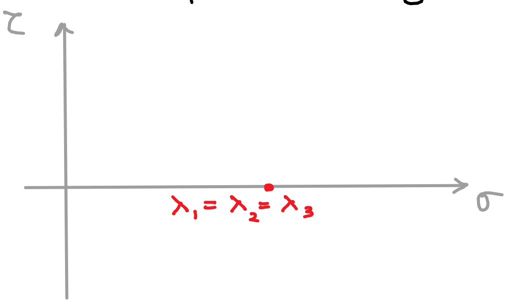
(the radius of largest, outer Mohr's circle)

<u>Special case I</u>: Two repeated eigenvalues (say $\lambda_1 > \lambda_2 = \lambda_3$)

Then the circle corresponding to (λ_2, λ_3) shrinks to a point



<u>Special case II</u>: Three repeated eigenvalues ($\lambda_1 = \lambda_2 = \lambda_3$)



Whole region shrinks to a point