8.5 Virtual Work

Consider a mass attached to a spring and pulled by an applied force F_{apl}, Fig. 8.5.1a. When the mass is in equilibrium, $F_{spr} + F_{apl} = 0$, where $F_{spr} = -kx$ is the spring force with x the distance from the spring reference position.

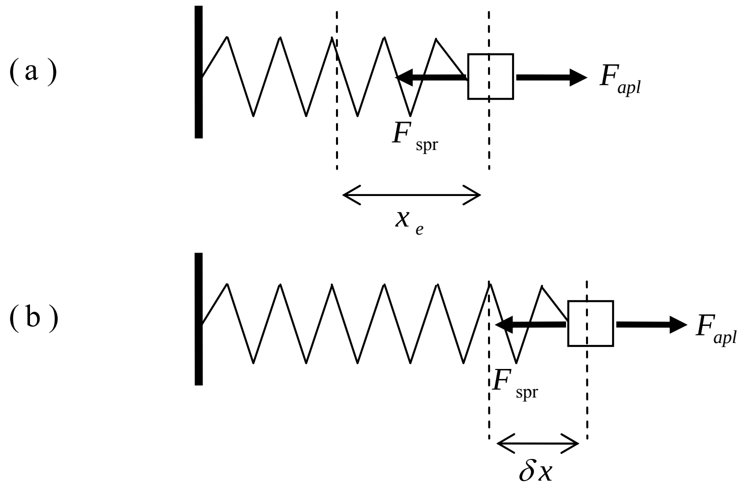


Figure 8.5.1: a force extending an elastic spring; (a) block in equilibrium, (b) block not at its equilibrium position

In order to develop a number of powerful techniques based on a concept known as **virtual work**, imagine that the mass is not in fact at its equilibrium position but at an (incorrect) non-equilibrium position $x + \delta x$, Fig. 8.5.1b. The imaginary displacement δx is called a **virtual displacement**. Define the *virtual work* δW done by a force to be the equilibrium force times this small imaginary displacement δx. It should be emphasized that virtual work is not real work – no work has been performed since δx is not a real displacement which has taken place; this is more like a "thought experiment". The virtual work of the spring force is then $\delta W_{spr} = F_{spr}\delta x = -kx\delta x$. The virtual work of the applied force is $\delta W_{apl} = F_{apl}\delta x$. The total virtual work is

$$\delta W = \delta W_{spr} + \delta W_{apl} = \left(-kx + F_{apl}\right)\delta x \tag{8.5.1}$$

There are two ways of viewing this expression. First, if the system is in equilibrium ($-kx + F_{apl} = 0$) then the virtual work is zero, $\delta W = 0$. Alternatively, if the virtual work is zero then, since δx is arbitrary, the system must be in equilibrium. Thus the virtual work idea gives one an alternative means of determining whether a system is in equilibrium.

The symbol δ is called a **variation** so that, for example, δx is a *variation in the displacement* (from equilibrium).

Virtual work is explored further in the following section.

8.5.1 Principle of Virtual Work: a single particle

A particle of mass m is acted upon by a number of forces, $\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_N$, Fig. 8.5.2. Suppose the particle undergoes a virtual displacement $\delta\mathbf{u}$; to reiterate, these impressed forces $\mathbf{f}_i$ do not *cause* the particle to move, one imagines it to be incorrectly positioned a little away from the true equilibrium position.

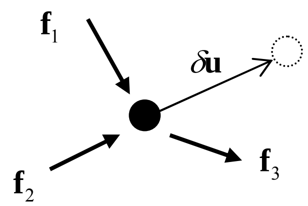


Figure 8.5.2: a particle in equilibrium under the action of a number of forces

If the particle is moving with an acceleration **a**, the quantity $-m\mathbf{a}$ is treated as an inertial force. The total virtual work is then (each term here is the dot product of two vectors)

$$\delta W = \left(\sum_{i=1}^{N} \mathbf{f}_i - m\mathbf{a} \right) \cdot \delta\mathbf{u} \tag{8.5.2}$$

Now *if* the particle is in equilibrium by the action of the effective (impressed plus inertial) force, then

$$\delta W = 0 \tag{8.5.3}$$

This can be expressed as follows:

The principle of virtual work (or **principle of virtual displacements**) **I**: if a particle is in equilibrium under the action of a number of forces (including the inertial force) the total work done by the forces for a virtual displacement is zero

Alternatively, one can define the external virtual work $\delta W_{\text{ext}} = \sum \mathbf{f}_i \cdot \delta\mathbf{u}$ and the virtual kinetic energy $\delta K = m\mathbf{a} \cdot \delta\mathbf{u}$ in which case the principle takes the form $\delta W_{\text{ext}} = \delta K$ (compare with the work-energy principle, Eqn. 8.1.10).

In the above, the principle of virtual work was derived using Newton's second law. One could just as well regard the principle of virtual work as the fundamental principle and from it derive the conditions for equilibrium. In this case one can say that[1]

[1] note the word *any* here: this must hold for *all* possible virtual displacements, for it will always be possible to find one virtual displacement which is perpendicular to the resultant of the forces, so that $\left(\sum \mathbf{f}\right) \cdot \delta\mathbf{u} = 0$ even though $\sum \mathbf{f}$ is not necessarily zero

The principle of virtual work (or **principle of virtual displacements**) **II**:
a particle is in equilibrium under the action of a system of forces (including the inertial force) if the total work done by the forces is zero for any virtual displacement of the particle

Constraints

In many practical problems, the particle will usually be constrained to move in only certain directions. For example consider a ball rolling over a table, Fig. 8.5.3. If the ball is in equilibrium then all the forces sum to zero, $\mathbf{R} + \sum \mathbf{f} - m\mathbf{a} = \mathbf{0}$, where one distinguishes between the non-reaction forces $\mathbf{f}_i$ and the reaction force **R**. If the virtual displacement $\delta\mathbf{u}$ is such that the constraint is not violated, that is the ball is not allowed to go "through" the table, then $\delta\mathbf{u}$ and **R** are perpendicular, the virtual work done by the reaction force is zero and $\delta W = \left(\sum \mathbf{f} - m\mathbf{a}\right) \cdot \delta\mathbf{u} = 0$. This is one of the benefits of the principle of virtual work; one does not need to calculate the forces of constraint **R** in order to determine the forces $\mathbf{f}_i$ which maintain the particle in equilibrium.

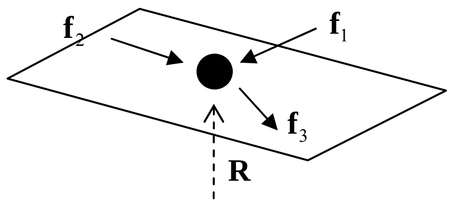


Figure 8.5.3: a particle constrained to move over a surface

The term **kinematically admissible displacement** is used to mean one that does not violate the constraints, and hence one arrives at the version of the principle which is often used in practice:

The principle of virtual work (or **principle of virtual displacements**) **III**:
a particle is in equilibrium under the action of a system of forces (including the inertial force) if the total work done by the forces (excluding reaction forces) is zero for any kinematically admissible virtual displacement of the particle

Whether one uses a kinematically admissible virtual displacement and so disregard reaction forces, or permit a virtual displacement that violates the constraint conditions will usually depend on the problem at hand. In this next example, use is made of a kinematically inadmissible virtual displacement.

Example

Consider a rigid bar of length L supported at its ends and loaded by a force F a distance a from the left hand end, Fig. 8.5.3a. Reaction forces R_A, R_C act at the ends. Let point C undergo a virtual displacement δu. From similar triangles, the displacement at B is $(a/L)\delta u$. End A does not move and so no virtual work is performed there. The total virtual work is

$$\delta W = R_C \delta u - F \frac{a}{L} \delta u \tag{8.5.4}$$

Note the minus sign here – the displacement at B is in a direction opposite to that of the action of the load and hence the work is negative. The beam is in equilibrium when $\delta W = 0$ and hence $R_C = aF / L$.

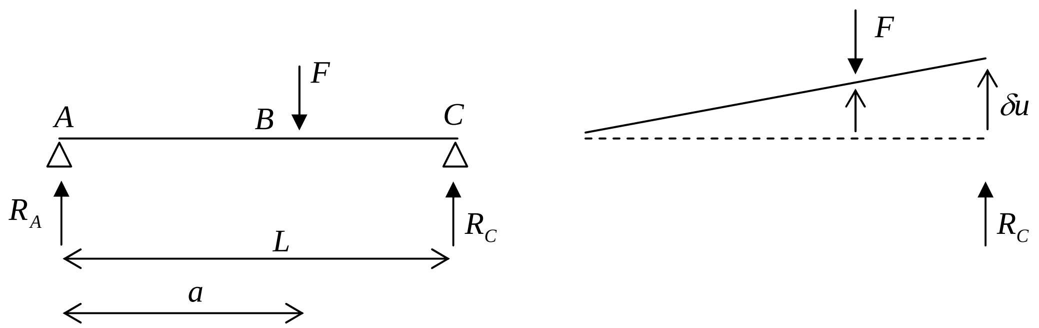


Figure 8.5.3: a loaded rigid bar; (a) bar geometry, (b) a virtual displacement at end C

■

8.5.2 Principle of Virtual Work: deformable bodies

A deformable body can be imagined to undergo virtual displacements (not necessarily the same throughout the body). Virtual work is done by the externally applied forces – **external virtual work** – and by the internal forces – **internal virtual work**. Looking again at the spring problem of Fig. 8.5.1, the external virtual work is $\delta W_{ext} = F_{apl} \delta x$ and, considering the spring force to be an "internal" force, the internal virtual work is $\delta W_{\text{int}} = -kx\delta x$. This latter virtual work can be re-written as $\delta W_{\text{int}} = -\delta U$ where δU is the virtual potential energy change which occurs when the spring is moved a distance δx (keeping the spring force constant).

In the same way, the internal virtual work of an elastic body is the (negative of the) virtual strain energy and the principle of virtual work can be expressed as

$$\boxed{\delta W_{ext} = \delta U} \quad \textbf{Principle of Virtual Work for an Elastic Body} \tag{8.5.4}$$

The principle can be extended to accommodate dissipation (energy loss), but only elastic materials will be examined here.

The virtual strain energy for a uniaxial rod is derived next.

8.5.3 Virtual Strain Energy for a Uniaxially Loaded Bar

In what follows, to distinguish between the strain energy and the displacement, the former will now be denoted by *w* and the latter by *u*.

Consider a uniaxial bar which undergoes strains ε. The strain is the unit change in length and, considering an element of length dx, Fig. 8.5.4a, the strain is

$$\varepsilon = \frac{[\Delta x + u(x+\Delta x) - u(x)] - \Delta x}{\Delta x} = \frac{du}{dx} \tag{8.5.5}$$

in the limit as $\Delta x \to 0$. With $dw = \sigma d\varepsilon$, the strain energy density is

$$w = \frac{1}{2}\sigma\varepsilon = \frac{1}{2}E\varepsilon^2 = \frac{1}{2}E\left(\frac{du}{dx}\right)^2 \tag{8.5.6}$$

and the strain energy is

$$U = \int_v \frac{1}{2}E\left(\frac{du}{dx}\right)^2 dV = \int_0^L \frac{EA}{2}\left(\frac{du}{dx}\right)^2 dx \tag{8.5.7}$$

This is the actual strain energy change when the bar undergoes actual strains ε. For the simple case of constant *A* and *L* and constant strain $du/dx = \Delta/L$ where Δ is the elongation of the bar, Eqn. 8.5.7 reduces to $U = AE\Delta^2/2L$ (equivalent to Eqn. 8.2.2).

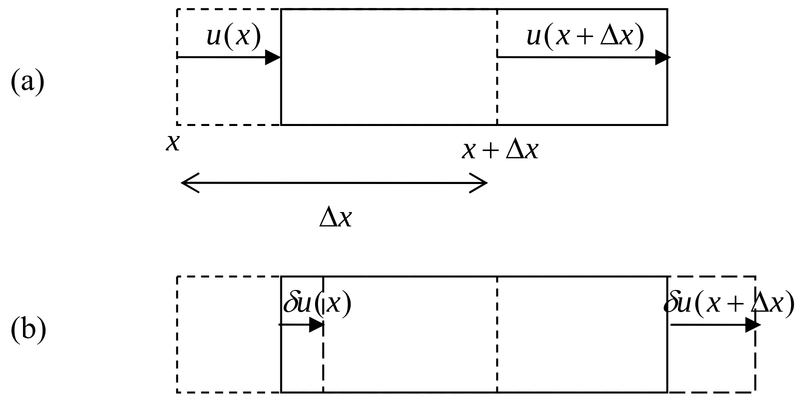


Figure 8.5.4: element undergoing actual and virtual displacements; (a) actual displacements, (b) virtual displacements

It will now be shown that the internal virtual work done as material particles undergo virtual displacements δu is given by δU, with *U* given by Eqn. 8.5.7.

Consider an element to "undergo" virtual displacements δu, Fig. 8.5.4b, which are, by definition, *measured from the actual displacements*. The virtual displacements give rise to **virtual strains**:

$$\delta\varepsilon = \frac{\delta u(x+\Delta x) - \delta u(x)}{\Delta x} = \frac{d(\delta u)}{dx} \tag{8.5.8}$$

again in the limit as $\Delta x \to 0$. Since $\delta\varepsilon = \delta(du/dx)$, it follows that

$$\delta\left(\frac{du}{dx}\right)=\frac{d(\delta u)}{dx} \tag{8.5.9}$$

In other words, the variation of the derivative is equal to the derivative of the variation[2].

One other result is needed before calculating the internal virtual work. Consider a function of the displacement, $f(u)$. The variation of *f* when *u* undergoes a virtual displacement is, by definition,

$$\delta f \equiv f(u+\delta u)-f(u)=\frac{f(u+\delta u)-f(u)}{\delta u}\delta u=\frac{df}{du}\delta u \tag{8.5.10}$$

now in the limit as the virtual displacement $\delta u \to 0$. From this one can write

$$\delta\left[\left(\frac{du}{dx}\right)^2\right]=2\left(\frac{du}{dx}\right)\delta\left(\frac{du}{dx}\right) \tag{8.5.11}$$

The stress σ applied to the surface of the element under consideration is an "external force". The internal force is the equal and opposite stress on the other side of the surface inside the element. The internal virtual work (per unit volume) is then $\delta W=-\sigma\delta\varepsilon$. Since σ is the *actual* stress, unaffected by the virtual straining,

$$\delta W=-E\varepsilon\delta\varepsilon=-E\left(\frac{du}{dx}\right)\delta\left(\frac{du}{dx}\right)=-\frac{1}{2}E\delta\left(\frac{du}{dx}\right)^2=-\delta\left[\frac{1}{2}E\left(\frac{du}{dx}\right)^2\right] \tag{8.5.12}$$

since the Young's modulus is unaffected by any virtual displacement. The total work done is then

$$\delta W_{\text{int}}=-\delta\int_v \frac{1}{2}E\left(\frac{du}{dx}\right)^2 dV \tag{8.5.13}$$

which, comparing with Eqn. 8.5.7, is the desired result, $\delta W_{\text{int}}=-\delta U$.

Example

Two rods with cross sectional areas A_1, A_2, lengths L_1, L_2 and Young's moduli E_1, E_2 and joined together with the other ends fixed, as shown in Fig. 8.5.5. The rods are subjected to a force *P* where they meet. As the rods elongate/contract, the strain is simply $\varepsilon=u_B/L$, where u_B is the displacement of the point at which the force is applied. The total elastic strain energy is, from Eqn. 8.5.7,

[2] this holds in general for any function; manipulations with variations form a part of a branch of mathematics known as the **Calculus of Variations**, which is concerned in the main with minima/maxima problems

$$U = \frac{E_1 A_1}{2L_1} u_B^2 + \frac{E_2 A_2}{2L_2} u_B^2 \tag{8.5.14}$$

Introduce now a virtual displacement δu_B at *B*. The external virtual work is $\delta W_{\text{ext}} = P \delta u_B$. The principle of virtual work, Eqn. 8.5.4, states that

$$P \delta u_B = \delta \left\{ \left(\frac{E_1 A_1}{2L_1} + \frac{E_2 A_2}{2L_2} \right) u_B^2 \right\} \tag{8.5.15}$$

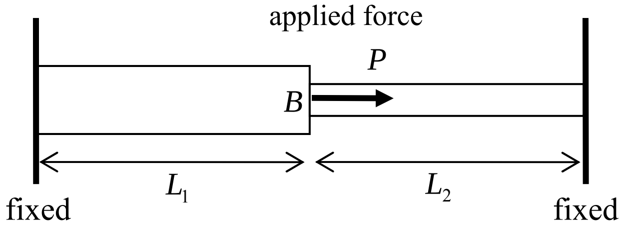


Figure 8.8.5: two rods subjected to a force *P*

From relation 8.5.10,

$$P \delta u_B = \left(\frac{E_1 A_1}{L_1} + \frac{E_2 A_2}{L_2} \right) u_B \delta u_B \tag{8.5.16}$$

The virtual displacement δu_B is arbitrary and so can be cancelled out, giving the result

$$u_B = P \left(\frac{E_1 A_1}{L_1} + \frac{E_2 A_2}{L_2} \right)^{-1} \tag{8.5.17}$$

from which the strains and hence stresses can be evaluated. Note that the reaction forces were not involved in this solution method.

■

8.5.4 Virtual Strain Energy for a Beam

The strain energy in a beam is given by Eqn. 8.2.7, *viz*.

$$U = \int_0^L \frac{M^2}{2EI} dx \tag{8.5.18}$$

Using the moment-curvature relation 7.4.37, $M = EI\left(d^2 v / dx^2\right)$, where *v* is the deflection of the beam,

$$U = \int_0^L \frac{EI}{2}\left(\frac{d^2 v}{dx^2}\right)^2 dx \qquad (8.5.19)$$

and the virtual strain energy is

$$\delta U = \delta \int_0^L \frac{EI}{2}\left(\frac{d^2 v}{dx^2}\right)^2 dx \qquad (8.5.20)$$

It is not easy to analyse problems using this expression and the principle of virtual work directly, but this expression will be used in the next section in conjunction with the related principle of minimum potential energy.

8.5.5 Problems

1. Consider a uniaxial bar of length *L* with constant cross section *A* and Young's modulus *E*, fixed at one end and subjected to a force *P* at the other. Use the principle of virtual work to show that the displacement at the loaded end is $u = PL/EA$.

2. Consider a uniaxial bar of length *L*, cross sectional area *A* and Young's modulus *E*. What factor of *EAL* is the strain energy when the displacements in the bar are $u = 10^{-3}x$, with *x* measured from one end of the bar? What is the internal virtual work for a virtual displacement $\delta u = 10^{-5}x$? For a constant virtual displacement along the bar?

3. A rigid bar rests upon three columns, a central column with Young's modulus 100GPa and two equidistant outer columns with Young's moduli 200GPa. The columns are of equal length 1m and cross-sectional area 1cm^2. The rigid bar is subjected to a downward force of 10kN. Use the principle of virtual work to evaluate the vertical displacement downward of the rigid bar.

4. Re-solve problem 3 from §8.2.6 using the principle of virtual displacements.

8.6 The Principle of Minimum Potential Energy

The **principle of minimum potential energy** follows directly from the principle of virtual work (for elastic materials).

8.6.1 The Principle of Minimum Potential Energy

Consider again the example given in the last section; in particular re-write Eqn. 8.5.15 as

$$\delta\left\{Pu_B - \left(\frac{E_1A_1}{2L_1} + \frac{E_2A_2}{2L_2}\right)u_B^2\right\} = 0 \tag{8.6.1}$$

The quantity inside the curly brackets is defined to be the **total potential energy** of the system, Π, and the equation states that the variation of Π is zero – that this quantity does not vary when a virtual displacement is imposed:

$$\delta\Pi = 0 \tag{8.6.2}$$

The total potential energy as a function of displacement *u* is sketched in Fig. 8.6.1. With reference to the figure, Eqn. 8.6.2 can be interpreted as follows: the total potential energy attains a stationary value (maximum or minimum) at the *actual* displacement (u_1); for example, $\delta\Pi \neq 0$ for an incorrect displacement u_2. Thus the solution for displacement can be obtained by finding a stationary value of the total potential energy. Indeed, it can be seen that the quantity inside the curly brackets in Fig. 8.6.1 attains a minimum for the solution already derived, Eqn. 8.5.17.

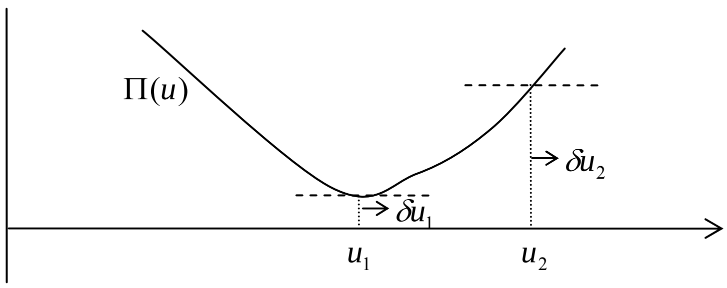


Figure 8.6.1: the total potential energy of a system

To generalise, define the "potential energy" of the applied loads to be $\delta V = -\delta W_{ext}$ so that

$$\delta\Pi = \delta U + \delta V \tag{8.6.3}$$

The external loads must be conservative, precluding for example any sliding frictional loading. Taking the total potential energy to be a function of displacement *u*, one has

$$\delta\Pi = \frac{d\Pi(u)}{du}\delta u = 0 \tag{8.6.4}$$

Thus of all possible displacements *u* satisfying the loading and boundary conditions, the actual displacement is that which gives rise to a stationary point $d\Pi / du = 0$ and the problem reduces to finding a stationary value of the total potential energy $\Pi = U + V$.

Stability

To be precise, Eqn. 8.6.2 only demands that the total potential energy has a stationary point, and in that sense it is called the **principle of stationary potential energy**. One can have a number of stationary points as sketched in Fig. 8.6.2. The true displacement is one of the stationary values u_1, u_2, u_3.

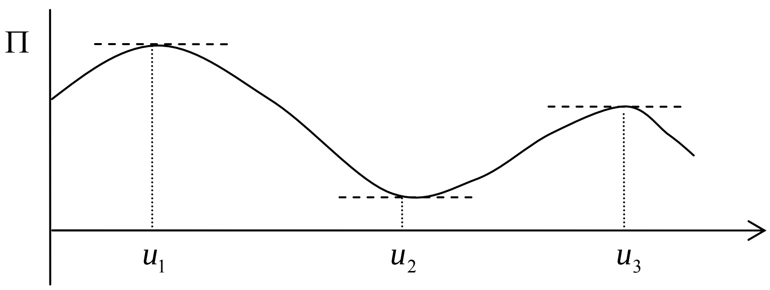


Figure 8.6.2: the total potential energy of a system

Consider the system with displacement u_2. If an external force acts to give the particles of the system some small initial velocity and hence kinetic energy, one has $0 = \Delta\Pi + \Delta K$. The particles will now move and so the displacement u_2 changes. Since Π is a minimum there it must increase and so the kinetic energy must decrease, and so the particles remain close to the equilibrium position. For this reason u_2 is defined as a **stable** equilibrium point of the system. If on the other hand the particles of the body were given small initial velocities from an initial displacement u_1 or u_3, the kinetic energy would increase dramatically; these points are called **unstable** equilibrium points. Only the state of stable equilibrium is of interest here and the principle of stationary potential energy in this case becomes the principle of minimum potential energy.

8.6.2 The Rayleigh-Ritz Method

In applications, the principle of minimum potential energy is used to obtain *approximate* solutions to problems which are otherwise difficult or, more usually, impossible to solve exactly. It forms one basis of the **Finite Element Method** (FEM), a general technique for solving systems of equations which arise in complex mechanics problems.

Example

Consider a uniaxial bar of length *L*, young's modulus *E* and varying cross-section $A = A_0(1 + x / L)$, fixed at one end and subjected to a force *F* at the other. The true solution for displacement to this problem can be shown to be $u = (FL/EA_0)\ln(1 + x/L)$. To see how this might be approximated using the principle, one writes

$$\Pi = U + V = \frac{1}{2}\int_0^L EA\left(\frac{du}{dx}\right)^2 dx - Fu\big|_{x=L} \tag{8.6.5}$$

First, substituting in the exact solution leads to

$$\Pi = \frac{EA_0}{2}\int_0^L (1 + x/L)\left(\frac{F}{EA_0}\frac{1}{1 + x/L}\right)^2 dx - F\frac{FL}{EA_0}\ln 2 = -\frac{\ln 2}{2}\frac{F^2 L}{EA_0} \tag{8.6.6}$$

According to the principle, any other displacement solution (which satisfies the displacement boundary condition $u(0) = 0$) will lead to a greater potential energy Π.

Suppose now that the solution was unknown. In that case an estimate of the solution can be made in terms of some unknown parameter(s), substituted into Eqn. 8.6.5, and then minimised to find the parameters. This procedure is known as the **Rayleigh Ritz method**. For example, let the guess, or **trial function**, be the linear function $u = \alpha + \beta x$. The boundary condition leads to $\alpha = 0$. Substituting $u = \beta x$ into Eqn. 8.6.5 leads to

$$\Pi = \frac{1}{2}EA_0\beta^2\int_0^L (1 + x/L)dx - F\beta L = \frac{3}{4}EA_0 L\beta^2 - F\beta L \tag{8.6.7}$$

The principle states that $\delta\Pi = (d\Pi/d\beta)\delta\beta = 0$, so that

$$\frac{d\Pi}{d\beta} = \frac{3}{2}EA_0 L\beta - FL = 0 \quad \to \quad \beta = \frac{2F}{3EA_0} \quad \to \quad u = \frac{2Fx}{3EA_0} \tag{8.6.8}$$

The exact and approximate Ritz solution are plotted in Fig. 8.6.3.

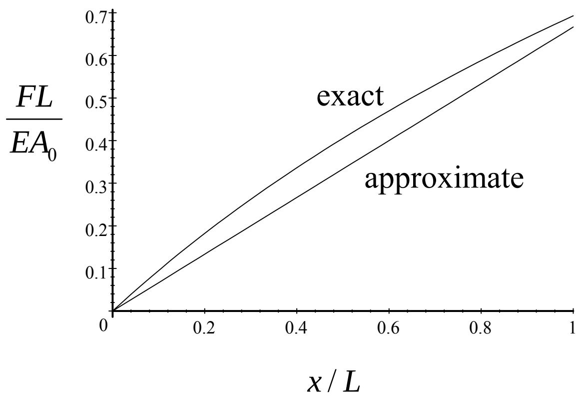


Figure 8.6.3: exact and (Ritz) approximate solution for axial problem

The total potential energy due to this approximate solution $2Fx/3EA_0$ is, from Eqn. 8.6.5,

$$\Pi = -\frac{1}{3}\frac{F^2 L}{EA_0} \tag{8.6.9}$$

which is indeed greater than the minimum value Eqn. 8.6.6 ($\approx -0.347F^2 L/EA_0$).

■

The accuracy of the solution 8.6.9 can be improved by using as the trial function a quadratic instead of a linear one, say $u = \alpha + \beta x + \gamma x^2$. Again the boundary condition leads to $\alpha = 0$. Then $u = \beta x + \gamma x^2$ and there are now two unknowns to determine. Since Π is a function of two variables,

$$\delta\Pi(\beta,\gamma) = \frac{\partial \Pi}{\partial \beta}\delta\beta + \frac{\partial \Pi}{\partial \gamma}\delta\gamma = 0 \tag{8.6.10}$$

and the two unknowns can be obtained from the two conditions

$$\frac{\partial \Pi}{\partial \beta} = 0, \qquad \frac{\partial \Pi}{\partial \gamma} = 0 \tag{8.6.11}$$

Example

A beam of length *L* and constant Young's modulus *E* and moment of inertia *I* is supported at its ends and subjected to a uniform distributed force per length *f*. Let the beam undergo deflection $v(x)$. The potential energy of the applied loads is

$$V = -\int_0^L f v(x)dx \tag{8.6.12}$$

and, with Eqn. 8.5.19, the total potential energy is

$$\Pi = \frac{EI}{2}\int_0^L \left(\frac{d^2 v}{dx^2}\right)^2 dx - f\int_0^L v dx \tag{8.6.13}$$

Choose a quadratic trial function $v = \alpha + \beta x + \gamma x^2$. The boundary conditions lead to $v = \gamma x(x - L)$. Substituting into 8.6.13 leads to

$$\Pi = 2\gamma^2 EIL - f\gamma L^3/6 \tag{8.6.14}$$

With $\delta\Pi = (d\Pi/d\gamma)\delta\gamma = 0$, one finds that

$$\gamma = \frac{fL^2}{24EI} \quad \to \quad v(x) = -\frac{fL^3}{24EI}x + \frac{fL^2}{24EI}x^2 \tag{8.6.15}$$

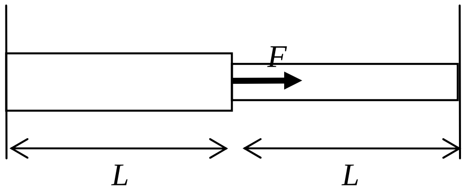
which compares with the exact solution

$$v(x) = -\frac{fL^3}{24EI}x + \frac{fL}{12EI}x^3 - \frac{f}{24EI}x^4 \tag{8.6.16}$$

■

8.6.3 Problems

1. Consider the statically indeterminate uniaxial problem shown below, two bars joined at $x = L$, built in at $x = 0$ and $x = 2L$, and subjected to a force *F* at the join. The cross-sectional area of the bar on the left is 2*A* and that on the right is *A*. Use the principle of minimum potential energy in conjunction with the Rayleigh-Ritz method with a trial displacement function of the form $u = \alpha + \beta x + \gamma x^2$ to approximate the exact displacement and in particular the displacement at $x = L$.



2. A beam of length *L* and constant Young's modulus *E* and moment of inertia *I* is supported at its ends and subjected to a uniform distributed force per length *f and* a concentrated force *P* at its centre. Use the principle of minimum potential energy in conjunction with the Rayleigh-Ritz method with a trial deflection $v = \alpha \sin(\pi x / L)$, to approximate the exact deflection.

3. Use the principle of minimum potential energy in conjunction with the Rayleigh-Ritz method with a trial solution $u = \alpha x$ to approximately solve the problem of axial deformation of an elastic rod of varying cross section, built in at one end and loaded by a uniform distributed force/length *f*, and a force *P* at the free end, as shown below. The cross sectional are is $A(x) = A_0(2 - x / L)$ and the length of the rod is *L*.