Principle of Conservation of Energy

In previous lecture, we introduce two concepts:

External work by a gradually applied load to a body causing deformation along the load

W_{ext}

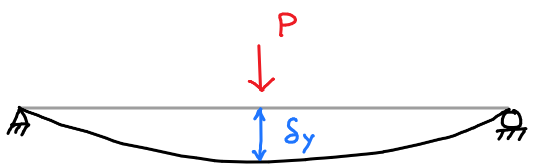
Internal stored strain energy caused by normal and shear stresses

U_i

The conservation of energy for a body with quasi-static deformation:

$$\boxed{W_{ext} = U_i}$$

So let's say if we wanted to find the vertical displacement δ_y of a beam under the gradual application of a load P



External work, $W_{ext} = \frac{1}{2} P \delta_y$

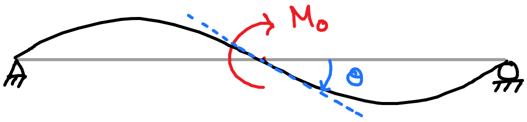
Internal strain energy would be caused by internal shear & bending moment caused by P.

$$U_i = \int_0^L \frac{M^2(x)}{2EI_z}\,dx + \int_0^L \frac{V^2(x)}{KGA}\,dx$$

By conservation of energy, $U_e = U_i$

$$\Rightarrow \quad \frac{1}{2} P \delta_y = \int_0^L \frac{M^2(x)}{2EI_z}\,dx + \int_0^L \frac{V^2(x)}{KGA}\,dx$$

If the beam was instead subjected to an external moment M_o, the moment would have caused a rotation θ at its point of application.

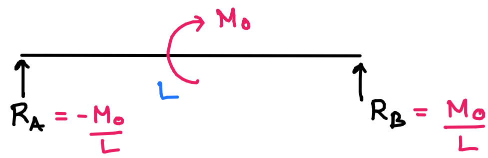


External work, $W_{ext} = \frac{1}{2} M_o \theta$

From conservation of energy,

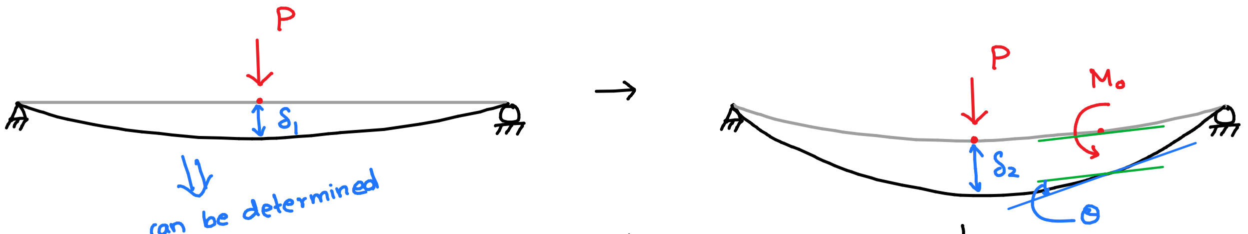
$$\frac{1}{2} M_o \theta = \int_0^L \frac{M^2(x)}{2EI} dx + \int_0^L \frac{V^2(x)}{2kGA} dx$$

Would shear force be present in the beam when only M_o is applied? Yes!



Using conservation of energy, one can find deflection or slope of a beam, or deformation of any body in general. However, the relation has very limited use because this method can be used to find deformation only when a single load is acting.

For more than one external load (force/moment), the external work for each loading would have its associated own unknown displacement. As such none of the displacements can be determined using a single equation $W_{ext} = U_i$



can be determined

$$\frac{1}{2} P \delta_1 + P \delta_2 + \frac{1}{2} M_o \theta = \int_0^L \frac{M^2(x)}{2EI} dx + \int_0^L \frac{V^2(x)}{2kGA} dx$$

2 unknowns, 1 eqn

8.2.4 Castigliano's Second Theorem

The work-energy method is the simplest of energy methods. A more powerful method is that based on **Castigliano's second theorem**, which can be used to solve problems involving *linear* elastic materials. As an introduction to Castigliano's second theorem, consider the case of uniaxial tension, where $U = P^2 L / 2EA$. The displacement through which the force moves can be obtained by a differentiation of this expression with respect to that force,

$$\frac{dU}{dP} = \frac{PL}{EA} = \Delta \qquad (8.2.24)$$

Similarly, for torsion of a circular bar, $U = T^2 L / 2GJ$, and a differentiation gives $dU / dT = TL / GJ = \phi$. Further, for bending of a beam it is also seen that $dU / dM = \theta$.

These are examples of Castigliano's theorem, which states that, provided the body is in equilibrium, *the derivative of the strain energy with respect to the force gives the displacement corresponding to that force, in the direction of that force.* When there is more than one force applied, then one takes the partial derivative. For example, if n independent forces $P_1, P_2, \dots, P_n$ act on a body, the displacement corresponding to the ith force is

$$\Delta_i = \frac{\partial U}{\partial P_i} \tag{8.2.25}$$

Before proving this theorem, here follow some examples.

Example

The beam shown in Fig. 8.2.11 is pinned at A, simply supported half-way along the beam at B and loaded at the end C by a force P and a moment M_0.

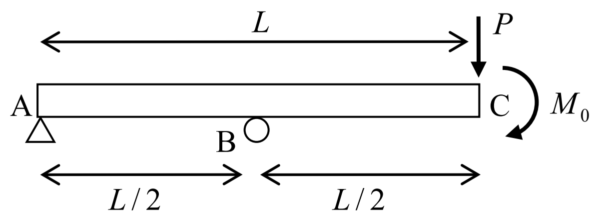


Figure 8.2.11: a beam subjected to a force and moment at C

The moment along the beam can be calculated from force and moment equilibrium,

$$M = \begin{cases} -Px - 2M_0 x / L, & 0 < x < L/2 \\ -M_0 - P(L - x), & L/2 < x < L \end{cases} \tag{8.2.26}$$

The strain energy stored in the bar (due to the flexural stresses only) is

$$\begin{aligned} U &= \frac{1}{2EI}\left\{\left(P + \frac{2M_0}{L}\right)^2 \int_0^{L/2} x^2 dx + \int_{L/2}^{L} \left(M_0 + P(L-x)\right)^2 dx\right\} \\ &= \frac{P^2 L^3}{24EI} + \frac{5PM_0 L^2}{24EI} + \frac{M_0^2 L}{3EI} \end{aligned} \tag{8.2.27}$$

In order to apply Castigliano's theorem, the strain energy is considered to be a function of the two external loads, $U = U(P, M_0)$. The displacement associated with the force P is then

$$\Delta_C = \frac{\partial U}{\partial P} = \frac{PL^3}{12EI} + \frac{5M_0L^2}{24EI} \tag{8.2.28}$$

The rotation associated with the moment is

$$\theta_C = \frac{\partial U}{\partial M_0} = \frac{5PL^2}{24EI} + \frac{2M_0L}{3EI} \tag{8.2.29}$$

■

Example

Consider next the beam of length *L* shown in Fig. 8.2.12, built in at both ends and loaded centrally by a force *P*. This is a statically indeterminate problem. In this case, the strain energy can be written as a function of the applied load and one of the unknown reactions.

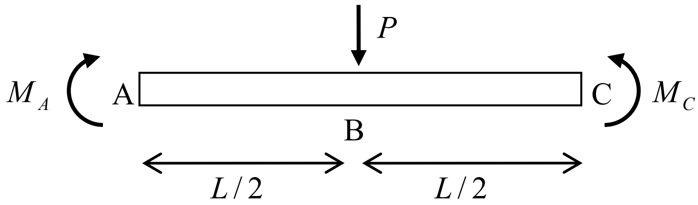


Figure 8.2.12: a statically indeterminate beam

First, the moment in the beam is found from equilibrium considerations to be

$$M = M_A + \frac{P}{2}x, \qquad 0 < x < L/2 \tag{8.2.30}$$

where M_A is the unknown reaction at the left-hand end. Then the strain energy in the left-hand half of the beam is

$$U = \frac{1}{2EI}\int_0^{L/2}\left(M_A + \frac{P}{2}x\right)^2 dx = \frac{P^2L^3}{192EI} + \frac{PM_AL^2}{16EI} + \frac{M_A^2L}{4EI} \tag{8.2.31}$$

The strain energy in the complete beam is double this:

$$U = \frac{P^2L^3}{96EI} + \frac{PM_AL^2}{8EI} + \frac{M_A^2L}{2EI} \tag{8.2.32}$$

Writing the strain energy as $U = U(P, M_A)$, the rotation at A is

$$\theta_A = \frac{\partial U}{\partial M_A} = \frac{PL^2}{8EI} + \frac{M_AL}{EI} \tag{8.2.33}$$

But $\theta_A = 0$ and so Eqn. 8.2.33 can be solved to get $M_A = -PL/8$. Then the displacement at the centre of the beam is

$$\Delta_B = \frac{\partial U}{\partial P} = \frac{PL^3}{48EI} + \frac{M_A L^2}{8EI} = \frac{PL^3}{192EI} \tag{8.2.34}$$

This is positive in the direction in which the associated force is acting, and so is downward.

■

Proof of Castigliano's Theorem

A proof of Castiligliano's theorem will be given here for a structure subjected to a single load. The load P produces a displacement Δ and the strain energy is $U = P\Delta/2$, Fig. 8.2.13. If an additional force dP is applied giving an additional deformation $d\Delta$, the additional strain energy is

$$dU = Pd\Delta + \frac{1}{2}dPd\Delta \tag{8.2.35}$$

If the load $P + dP$ is applied from zero in one step, the work done is $(P + dP)(\Delta + d\Delta)/2$. Equating this to the strain energy $U + dU$ given by Eqn. 8.2.35 then gives $Pd\Delta = \Delta dP$. Substituting into Eqn. 8.2.35 leads to

$$dU = \Delta dP + \frac{1}{2}dPd\Delta \tag{8.2.36}$$

Dividing through by dP and taking the limit as $d\Delta \to 0$ results in Castigliano's second theorem, $dU/dP = \Delta$.

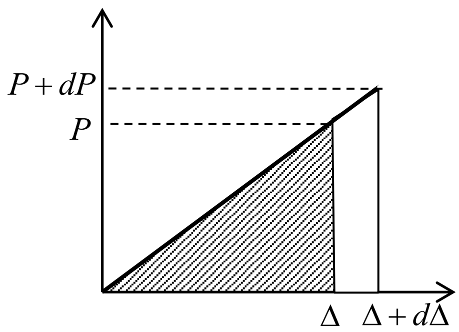


Figure 8.2.13: force-displacement curve

In fact, dividing Eqn. 8.2.35 through by $d\Delta$ and taking the limit as $d\Delta \to 0$ results in **Castigliano's first theorem**, $dU/d\Delta = P$. It will be shown later that this first theorem, unlike the second, in fact holds also for the case when the elastic material is *non-linear*.

Therefore, we have:

$$dU_i = dP_j \, \delta_j$$

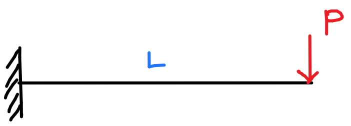
$$\Rightarrow \boxed{\delta_j = \frac{dU_i}{dP_j}}$$

Castigliano's theorem states that the displacement at a pt in the body is equal to the first derivative of the strain energy in the body w.r.t. a force acting at that point and along the direction of the displacement.

Using a dummy force/moment

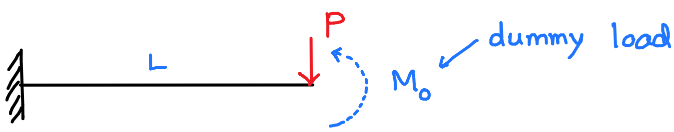
If we are interested to find displacement at a point in the body where there is no corresponding applied load, a dummy load is introduced and then Castigliano's theorem is applied.

Ex1: Find the rotation of the free end of the cantilever beam



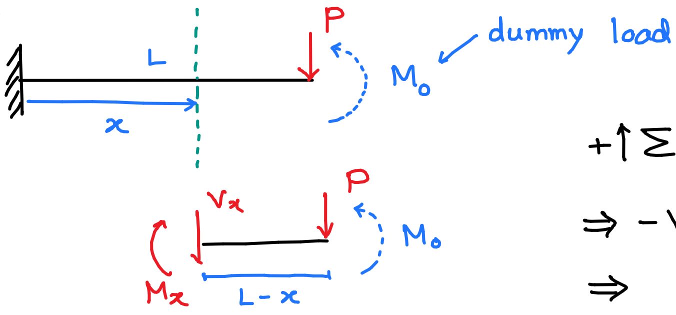
Note that the corresponding load for getting rotation at the end is a moment applied at the end

1. Apply a dummy load P_o or M_o the location of required deflection



2. Obtain the strain energy of the body, U_i
3. Apply Castigliano's theorem, i.e. $\frac{\partial U_i}{\partial P_o}$ or $\frac{\partial U_i}{\partial M_o}$
4. Finally, set $P_o = 0$ (or $M_o = 0$) in the expression for deflection.

Ex 1



$$+\uparrow \sum F_y = 0$$

$$\Rightarrow -V(x) - P = 0$$

$$\Rightarrow V(x) = -P$$

$$\circlearrowleft + \sum M_{\text{left end}} = 0$$

$$\Rightarrow -M(x) - P(L-x) + M_o = 0$$

$$\Rightarrow M(x) = M_o - P(L-x)$$

The axial force and torque are zero. So the strain energy would be due to bending moment and shear force.

$$U_i = \int_0^L \frac{M^2(x)}{2EI}\,dx + \int_0^L \frac{V^2(x)}{2kGA}\,dx$$

$$= \int_0^L \left[\frac{(M_o - P(L-x))^2}{2EI} + \frac{(-P)^2}{2kGA}\right] dx$$

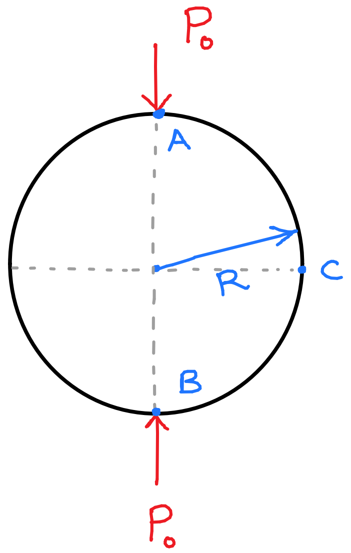
$$\theta = \frac{\partial U_i}{\partial M_o} = \int_0^L \left[\frac{\partial}{\partial M_o}\frac{(M_o - P(L-x))^2}{2EI} + \frac{\partial}{\partial M_o}\frac{P^2}{2kGA}\right] dx$$

$$= \int_0^L \frac{\cancelto{0}{M_o} - P(L-x)}{EI}\,dx$$

$$= -\frac{P}{EI}\int_0^L (L-x)\,dx = -\frac{P}{EI}\left(Lx - \frac{x^2}{2}\right)\Big|_0^L = (-)\frac{PL^2}{2EI}$$

the direction of rotation would be opposite of the direction of M_o assumed.

Ex 2 Curved beam



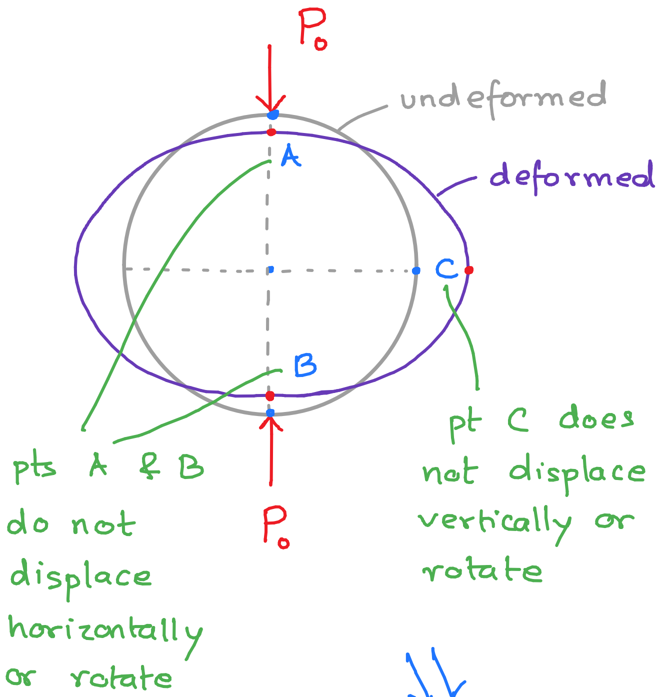
Consider a ring of radius R which is subjected to equal and opposite forces along its vertical diameter.

(a) By how much does A and B get closer to each other?

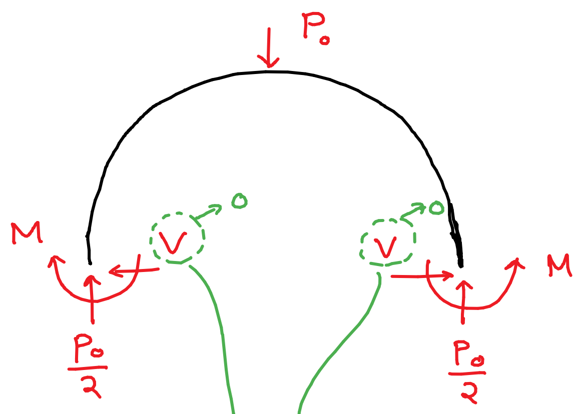
(b) By how much does point C move outward?

(c) What is the internal bending moment at point C?

Due to the symmetry of loading and that is geometry of the beam, the deformation of the beam will also be symmetric



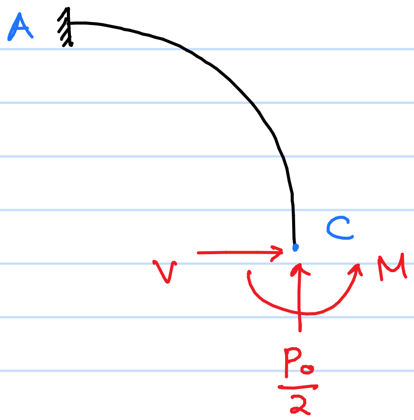
Let us divide the ring into two equal parts about the diameter line passing through C



we can look at one quarter of the ring for analysis

Due to symmetry of the top and lower half of beam deformation, the shear force should act either towards center or outwards for both halves (top & bottom) of beams
So, $V = 0$

One quarter of the ring



The clamping of point A implies that all displacements will be obtained relative to point A. So point C will now have vertical displacement equal and opposite to that at point A in the full ring problem.