8.1 Energy in Deforming Materials

There are many different types of **energy**: mechanical, chemical, nuclear, electrical, magnetic, etc. Energies can be grouped into **kinetic energies** (which are due to movement) and **potential energies** (which are stored energies – energy that a piece of matter has because of its position or because of the arrangement of its parts).

A rubber ball held at some height above the ground has (gravitational) potential energy. When dropped, this energy is progressively converted into kinetic energy as the ball's speed increases until it reaches the ground where all its energy is kinetic. When the ball hits the ground it begins to deform elastically and, in so doing, the kinetic energy is progressively converted into **elastic strain energy**, which is stored *inside* the ball. This elastic energy is due to the re-arrangement of molecules in the ball – one can imagine this to be very like numerous springs being compressed inside the ball. The ball reaches maximum deformation when the kinetic energy has been completely converted into strain energy. The strain energy is then converted back into kinetic energy, "pushing" the ball back up for the rebound.

Elastic strain energy is a potential energy – elastically deforming a material is in many ways similar to raising a weight off the ground; in both cases the potential energy is increased.

Similarly, **work** is done in stretching a rubber band. This work is converted into elastic strain energy within the rubber. If the applied stretching force is then slowly reduced, the rubber band will use this energy to "pull" back. If the rubber band is stretched and then released suddenly, the band will retract quickly; the strain energy in this case is converted into kinetic energy – and sound energy (the "snap").

When a small weight is placed on a large metal slab, the slab will undergo minute strains, too small to be noticed visually. Nevertheless, the metal behaves like the rubber ball and when the weight is removed the slab uses the internally stored strain energy to return to its initial state. On the other hand, a metal bar which is bent considerably, and then laid upon the ground, will not nearly recover its original un-bent shape. It has undergone *permanent* deformation. Most of the energy supplied has been lost; it has been converted into heat energy, which results in a very slight temperature rise in the bar. Permanent deformations of this type are accounted for by **plasticity theory**, which is treated in Chapter 11.

In any real material undergoing deformation, at least some of the supplied energy will be converted into heat. However, with the ideal elastic material under study in this chapter, it is assumed that *all* the energy supplied is converted into strain energy. When the loads are removed, the material returns to its precise initial shape and there is no energy loss; for example, a purely elastic ball dropped onto a purely elastic surface would bounce back up to the precise height from which it was released.

As a prelude to a discussion of the energy of elastic materials, some important concepts from elementary particle mechanics are reviewed in the following sections. It is shown that Newton's second law, the **principle of work and kinetic energy** and the **principle of conservation of mechanical energy** are equivalent statements; each can be derived from the other. These concepts are then used to study the energetics of elastic materials.

8.1.1 Work and Energy in Particle Mechanics

Work

Consider a force F which acts on a particle, causing it to move through a displacement *s*, the directions in which they act being represented by the arrows in Fig. 8.1.1a. The work *W* done by *F* is defined to be $Fs\cos\theta$ where θ is the angle formed by positioning the start of the *F* and *s* arrows at the same location with $0 \le \theta \le 180$. Work can be positive or negative: when the force and displacement are in the same direction, then $0 \le \theta \le 90$ and the work done is positive; when the force and displacement are in opposite directions, then $90 \le \theta \le 180$ and the work done is negative.

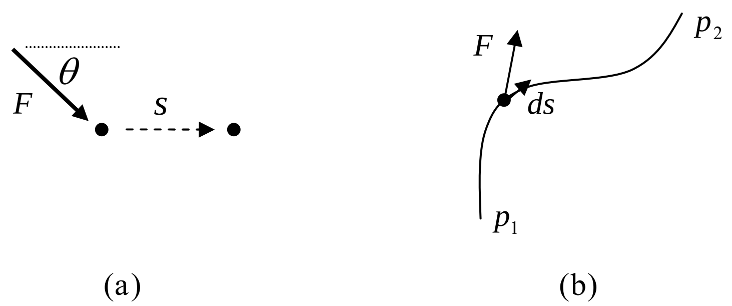


Figure 8.1.1: (a) force acting on a particle, which moves through a displacement *s*; (b) a varying force moving a particle along a path

Consider next a particle moving along a certain path between the points p_1, p_2 by the action of some force *F*, Fig. 8.1.1b. The work done is

$$W = \int_{p_1}^{p_2} F\cos\theta\, ds \qquad (8.1.1)$$

where s is the displacement. For motion along a straight line, so that $\theta = 0$, the work is $W = \int_{p_1}^{p_2} F ds$; if *F* here is *constant* then the work is simply *F* times the distance between p_1 to p_2 but, in most applications, *the force will vary* and an integral needs to be evaluated.

Conservative Forces

From Eqn. 8.1.1, the work done by a force in moving a particle through a displacement will in general *depend on the path* taken. There are many important practical cases, however, when the work is *independent* of the path taken, and simply depends on the initial and final positions, for example the work done in deforming elastic materials (see later) – these lead to the notion of a **conservative** (or **potential**) **force**. Looking at the one-dimensional case, a conservative force F_{con} is one which can always be written as the derivative of a function *U* (the minus sign will become clearer in what follows),

$$F_{\text{con}} = -\frac{dU}{dx}, \tag{8.1.2}$$

since, in that case,

$$W = \int_{p_1}^{p_2} F_{\text{con}}\, dx = -\int_{p_1}^{p_2} \frac{dU}{dx} dx = -\int_{p_1}^{p_2} dU = -\left(U(p_2) - U(p_1)\right) = -\Delta U \tag{8.1.3}$$

In this context, the function U is called the **potential energy** and ΔU is the change in potential energy of the particle as it moves from p_1 to p_2. If the particle is moved from p_1 to p_2 and then back to p_1, the net work done is zero and the potential energy U of the particle is that with which it started.

Potential Energy

The potential energy of a particle/system can be defined as follows:

Potential Energy: the work done in moving a system from some standard configuration to the current configuration

Potential energy has the following characteristics:
(1) The existence of a force field
(2) To move something in the force field, work must be done
(3) The force field is conservative
(4) There is some reference configuration
(5) The force field itself does negative work when another force is moving something against it
(6) It is recoverable energy

These six features are evident in the following example: a body attached to the coil of a spring is extended slowly by a force F, overcoming the spring (restoring) force F_{spr} (so that there are no accelerations and $F = -F_{\text{spr}}$ at all times), Fig. 8.1.2.

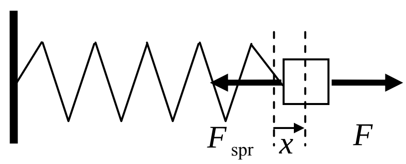


Figure 8.1.2: a force extending an elastic spring

Let the initial position of the block be x_0 (relative to the reference configuration, $x = 0$). Assuming the force to be proportional to deflection, $F = kx$, the work done by F in extending the spring to a distance x is

$$W = \int_{x_0}^{x} F dx = \int_{x_0}^{x} kx dx = \tfrac{1}{2}kx^2 - \tfrac{1}{2}kx_0^2 \equiv U(x) - U(x_0) = \Delta U \tag{8.1.4}$$

This is the work done to move something in the elastic spring "force field" and by definition is the potential energy (change in the body). The energy supplied in moving the body is said to be **recoverable** because the spring is ready to pull back and do the same amount of work.

The corresponding work done by the conservative spring force F_{spr} is

$$W_{\text{spr}} = -\int_{x_0}^{x} F dx = -\left(\tfrac{1}{2}kx^2 - \tfrac{1}{2}kx_0^2\right) \equiv -\Delta U \tag{8.1.5}$$

This work can be seen from the area of the triangles in Fig. 8.1.3: the spring force is zero at the equilibrium/reference position ($x = 0$) and increases linearly as x increases.

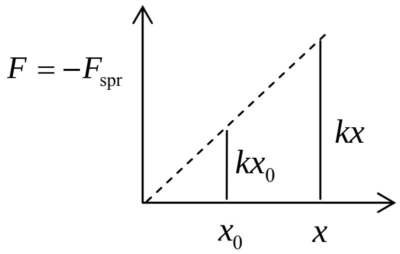


Figure 8.1.3: force-extension curve for a spring

■

The forces in this example depend on the amount by which the spring is stretched. This is similar to the potential energy stored in materials – the potential force will depend in some way on the separation between material particles (see below).

Also, from the example, it can be seen that an alternative definition for the potential energy U of a system is *the negative of the work done by a conservative force in moving the system from some standard configuration to the current configuration*.

In general then, the work done by a conservative force is related to the potential energy through

$$W_{con} = -\Delta U \tag{8.1.6}$$

Dissipative (Non-Conservative) Forces

When the forces are not conservative, that is, they are **dissipative**, one cannot find a universal function U such that the work done is the difference between the values of U at the beginning and end points – *one has to consider the path* taken by the particle and the work done will be different in each case. A general feature of non-conservative forces is that if one moves a particle and then returns it to its original position the net work done will not be zero. For example, consider a block being dragged across a rough surface,

Fig. 8.1.4. In this case, if the block slides over and back a number of times, the work done by the pulling force F keeps increasing, and the work done is not simply determined by the final position of the block, but by its complete path history. The energy used up in moving the block is dissipated as heat (the energy is **irrecoverable**).

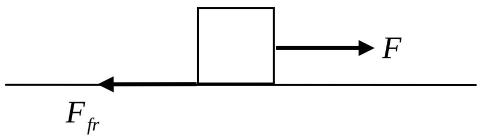


Figure 8.1.4: Dragging a block over a frictional surface

8.1.2 The Principle of Work and Kinetic Energy

In general, a mechanics problem can be solved using either Newton's second law or the principle of work and energy (which is discussed here). These are two different equations which basically say the same thing, but one might be preferable to the other depending on the problem under consideration. Whereas Newton's second law deals with *forces*, the work-energy principle casts problems in terms of *energy*.

The kinetic energy of a particle of mass m and velocity v is defined to be $K = \frac{1}{2}mv^2$. The rate of change of kinetic energy is, using Newton's second law $F = ma$,

$$\dot{K} = \frac{d}{dt}\left(\frac{1}{2}mv^2\right) = mv\frac{dv}{dt} = (ma)v = Fv \tag{8.1.7}$$

The change in kinetic energy over a time interval (t_0, t_1) is then

$$\Delta K = K_1 - K_0 = \int_{t_0}^{t_1} \frac{dK}{dt}dt = \int_{t_0}^{t_1} Fv\,dt \tag{8.1.8}$$

where K_0 and K_1 are the initial and final kinetic energies. The work done over this time interval is

$$W = \int_{W(t_0)}^{W(t_1)} dW = \int_{x(t_0)}^{x(t_1)} F\,dx = \int_{t_0}^{t_1} Fv\,dt \tag{8.1.9}$$

and it follows that

$$\boxed{W = \Delta K} \qquad \textbf{Work – Energy Principle} \tag{8.1.10}$$

One has the following:

The principle of work and kinetic energy:
the total work done by the external forces acting on a particle equals the change in kinetic energy of the particle

It is not a new principle of mechanics, rather a rearrangement of Newton's second law of motion (or one could have started with this principle, and derived Newton's second law).

The following example shows how the principle holds for conservative, dissipative and applied forces.

Example

A block of mass m is attached to a spring and dragged along a rough surface. It is dragged from left to right, Fig. 8.1.5. Three forces act on the block, the applied force F_{apl} (taken to be constant), the spring force F_{spr} and the friction force F_{fri} (assumed constant).

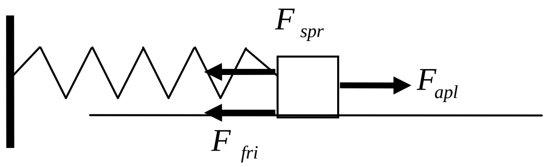


Figure 8.1.5: a block attached to a spring and dragged along a rough surface

Newton's second law, with $F_{spr} = kx$, leads to a standard non-homogeneous second order linear ordinary differential equation with constant coefficients:

$$m\frac{d^2x}{dt^2} = F_{apl} - F_{fri} - F_{spr} \tag{8.1.11}$$

Taking the initial position of the block to be x_0 and the initial velocity to be $\dot{x}_0$, the solution can be found to be

$$x(t) = \frac{F_{apl} - F_{fri}}{k} + \left(x_0 - \frac{F_{apl} - F_{fri}}{k} \right) \cos \omega t + \frac{\dot{x}_0}{\omega} \sin \omega t \tag{8.1.12}$$

where $\omega = \sqrt{k/m}$. The total work done W is the sum of the work done by the applied force W_{apl}, the work done by the spring force W_{spr} and that done by the friction force W_{fri}:

$$W = W_{apl} + W_{spr} + W_{fri} = F_{apl}(x - x_0) - \frac{1}{2}k\left(x^2 - x_0^2\right) - F_{fri}(x - x_0) \tag{8.1.13}$$

The change in kinetic energy of the block is

$$\Delta K = \frac{1}{2} m\left(\dot{x}^2 - \dot{x}_0^2\right) \tag{8.1.14}$$

Substituting Eqn. 8.1.12 into 8.1.13-14 and carrying out the algebra, one indeed finds that $W = \Delta K$:

$$W = W_{apl} + W_{spr} + W_{fri} = \Delta K \tag{8.1.15}$$

Now the work done by the spring force is equivalent to the negative of the potential energy change, so the work-energy equation (8.1.15) can be written in the alternative form[1]

$$W_{apl} + W_{fri} = \Delta U_{spr} + \Delta K \tag{8.1.16}$$

The friction force is dissipative – it leads to energy loss. In fact, the work done by the friction force is converted into heat which manifests itself as a temperature change in the block. Denoting this energy loss by (see Eqn. 8.1.13) $H_{fri} = F_{fri}(x - x_0)$, one has

$$W_{apl} - H_{fri} = \Delta U_{spr} + \Delta K \tag{8.1.17}$$

■

8.1.3 The Principle of Conservation of Mechanical Energy

In what follows, it is assumed that *there is no energy loss*, so that no dissipative forces act. Define the **total mechanical energy** of a body to be the sum of the kinetic and potential energies of the body. The work-energy principle can then be expressed in two different ways, for this special case:

1. The total work done by the external forces acting on a body equals the change in kinetic energy of the body:

$$W = W_{con} + W_{apl} = \Delta K \tag{8.1.18}$$

2. The total work done by the external forces acting on a body, exclusive of the conservative forces, equals the change in the total mechanical energy of the body

$$W_{\mathrm{apl}} = \Delta U + \Delta K \tag{8.1.19}$$

The special case where there are no external forces, or where all the external forces are conservative/potential, leads to $0 = \Delta U + \Delta K$, so that the mechanical energy is *constant*. This situation occurs, for example, for a body in free-fall {▲Problem 3} and for a freely oscillating spring {▲Problem 4}. Both forms of the work-energy principle can also be seen to apply for a spring subjected to an external force {▲Problem 5}.

[1] it is conventional to keep work terms on the left and energy terms on the right

The Principle of Conservation of Mechanical Energy

The **principle of conservation of energy** states that the total energy of a system remains constant – energy cannot be created or destroyed, it can only be changed from one form of energy to another.

The principle of conservation of energy in the case where there is no energy dissipation is called the **principle of conservation of mechanical energy** and states that, *if a system is subject only to conservative forces, its mechanical energy remains constant*; any system in which non-conservative forces act will inevitable involve non-mechanical energy (heat transfer).

So, when there are only conservative forces acting, one has

$$0 = \Delta U + \Delta K \tag{8.1.20}$$

or, equivalently,

$$K_f + U_f = K_i + U_i \tag{8.1.21}$$

where K_i, K_f are the initial and final kinetic energies and U_i, U_f are the initial and final potential energies.

Note that the principle of mechanical energy conservation is not a new separate law of mechanics, it is merely a re-expression of the work-energy principle (or of Newton's second law).

8.1.4 Deforming Materials

The discussion above which concerned particle mechanics is now generalized to that of a deforming material.

Any material consists of many molecules and particles, all interacting in some complex way. There will be a complex system of **internal forces** acting between the molecules, even when the material is in a natural (undeformed) equilibrium state. If **external forces** are applied, the material will deform and the molecules will move, and hence not only will work be done by the external forces, but *work will be done by the internal forces*. The work-energy principle in this case states that the total work done by the external and internal forces equals the change in kinetic energy,

$$W_{\text{ext}} + W_{\text{int}} = \Delta K \tag{8.1.22}$$

In the special case where no external forces act on the system, one has

$$W_{\text{int}} = \Delta K \tag{8.1.23}$$

which is a situation known as **free vibration**. The case where the kinetic energy is unchanging is

$$W_{\text{ext}} + W_{\text{int}} = 0 \tag{8.1.24}$$

and this situation is known as **quasi-static** (the quantities here can still depend on time).

The force interaction between the molecules can be grouped into:
(1) conservative internal force systems
(2) non-conservative internal force systems (or at least partly non-conservative)

Conservative Internal Forces

First, assuming a conservative internal force system, one can imagine that the molecules interact with each other in the manner of elastic springs. Suppose one could apply an external force to pull two of these molecules apart, as shown in Fig. 8.1.6.

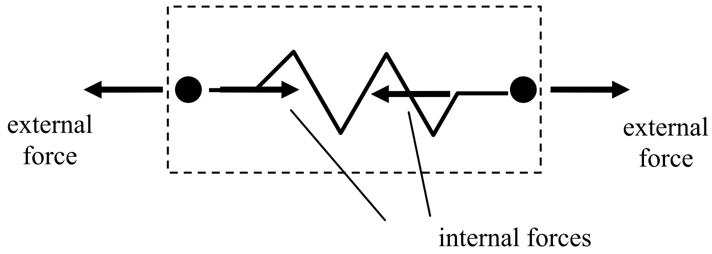


Figure 8.1.6: external force pulling two molecules/particles apart

In this ideal situation one can say that the work done by the external forces equals the change in potential energy plus the change in kinetic energy,

$$W_{\text{ext}} = \Delta U + \Delta K \tag{8.1.25}$$

The energy U in this case of deforming materials is called the **elastic strain energy**, the energy due to the molecular arrangement relative to some equilibrium position.

The free vibration case is now $0 = \Delta U + \Delta K$ and the quasi-static situation is $W_{\text{ext}} = \Delta U$.

Non-Conservative Internal Forces

Consider now another example of internal forces acting within materials, that of a polymer with long-chain molecules. If one could somehow apply an external force to a pair of these molecules, as shown in Fig. 8.1.7, the molecules would slide over each other. Frictional forces would act between the molecules, very much like the frictional force between the block and rough surface of Fig. 8.1.4. This is called **internal friction**. Assuming that the internal forces are dissipative, the external work cannot be written in terms of a potential energy, $W_{\text{ext}} \neq \Delta U + \Delta K$, since *the work done depends on the path taken*. One would have to calculate the work done by evaluating an integral.

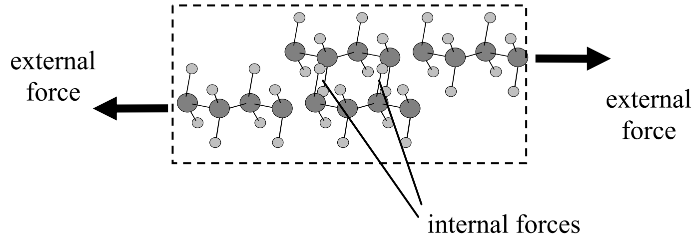


Figure 8.1.7: external force pulling two molecules/particles apart

Similar to Eqn. 8.1.17, however, the energy balance can be written as

$$W_{\mathrm{ext}} - H = \Delta K + \Delta U \tag{8.1.26}$$

Energy Methods

In the last lecture, we talked about deflection of beams subjected to transverse loads and moments. However, often, we may be interested not in the full deflection profile but only in the deflection at a specific point in the beam. For such cases, energy methods offers an easier alternative to finding at certain points.

External Work and Strain Energy

We first define the work caused by an external force/moment and show how to express this work in terms of a body's strain energy.

Work done by a force/moment

A force does work when the force acting at a point on the body undergoes a displacement in the same direction as the force

$$W_{ext} = F\,dx$$

displacement in the same direction as the force

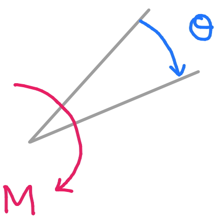
work done

If the total displacement is δ, then the external work done is:

$$W_{ext} = \int_0^{\delta} F\,dx$$

Similarly, a moment M does work when it causes a rotation $d\theta$ along its line of action. The external work done by moment due to a total rotation of θ is:

$$W_{ext} = \int_0^{\theta} M\,d\theta$$

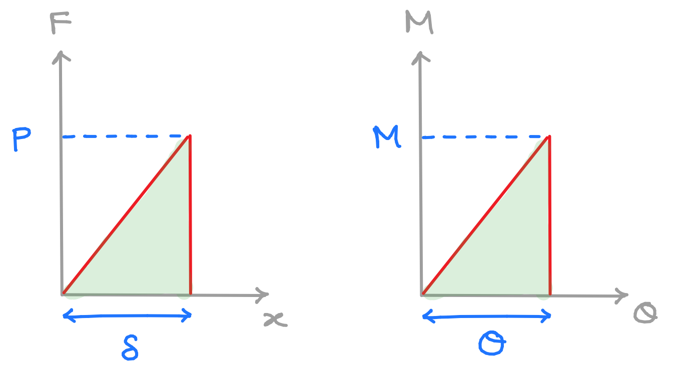


If a force/moment is gradually applied i.e. increased from zero to some value P (or M), with the corresponding deformation (displacement / rotation) increasing from 0 to a final value, then the work done in the case of linear elastic material:

$$W_{ext} = \frac{1}{2} F \delta$$

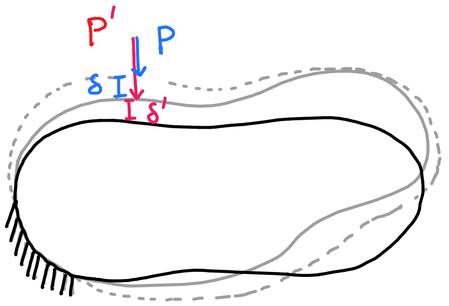
or,

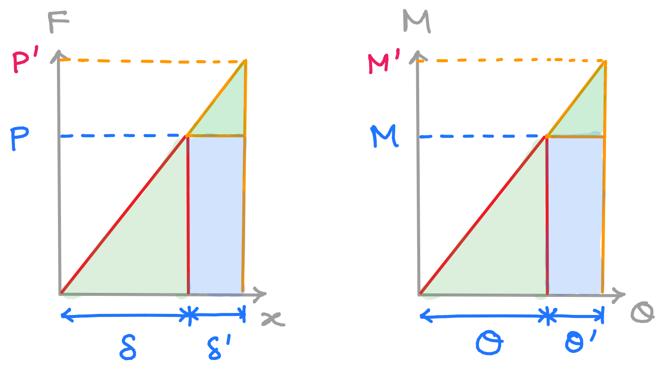
$$W_{ext} = \frac{1}{2} M \theta$$



Now, suppose a force P is already applied to the body at a point and another force P′ is applied gradually at the same point such that the body is displaced further by δ′, then the additional work done by P is

$$W'_{ext} = P \delta'$$





Similarly, for a moment that is already applied to the body and other loadings further rotate the body by an amt θ′, then the additional work done is : $W'_{ext} = M\theta'$

Strain Energy Density for Linear Elastic Bodies

When loads (forces/moments) are applied to a body, they will deform the material, and if no energy is lost in the form of heat or sound, the external work done by the loads will be converted into internal stored energy called the strain energy.

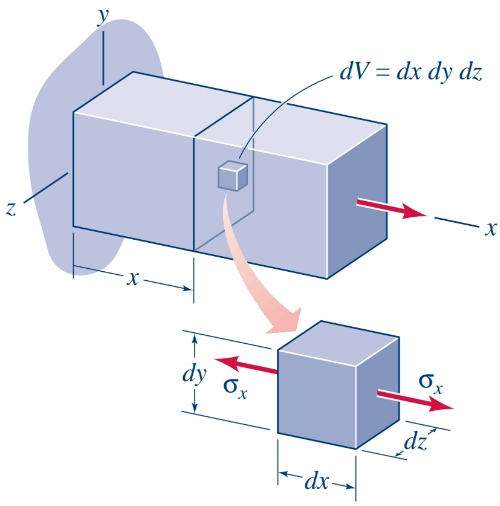
The strain energy per unit volume is called strain energy density.

Let dU_i be the strain energy stored in an elemental volume of a linear elastic body, like dV.

strain energy density

$$dU_i = \bar{u}\, dV$$

The total strain energy is obtained by integrating the dU_i's over the entire volume:



Total stored internal strain energy

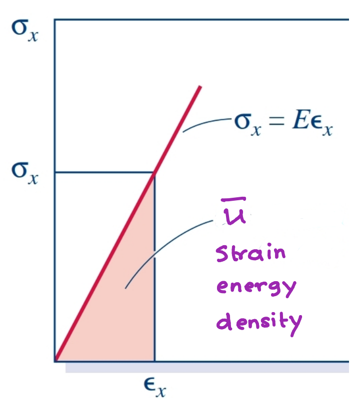
$$U_i = \int_V U_i = \int_V \bar{u}\, dV$$

For the uniaxial stress state shown in figure, we have

$$dU = \left(\frac{1}{2}\sigma_x\, dy\, dz\right)\left(\epsilon_x\, dx\right) = \bar{u}_{\sigma_x}\, dV$$

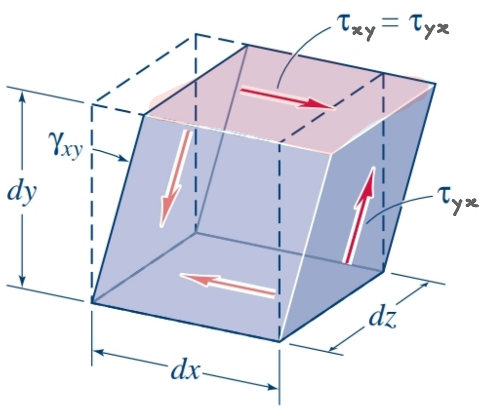
avg. force × disp.

where, $\bar{u}_{\sigma_x} = \frac{1}{2}\sigma_x \epsilon_x$ is the strain energy density for linear elastic deformation due to a uniaxial stress state σ_x. Since $\sigma_x = E\epsilon_x$ the strain energy density can be written as:



$$\bar{u}_{\sigma_x} = \frac{1}{2}\sigma_x \epsilon_x = \frac{1}{2}\frac{\sigma_x^2}{E} = \frac{1}{2}E\epsilon_x^2$$

So the area under the curve of a stress-strain diagram depicts the strain energy density (i.e. strain energy per unit volume)



Next consider an elemental volume subjected to only shear stress τ_{xy}. For this case, the strain energy would be

$$dU_i = \underbrace{\left(\frac{1}{2}\,\tau_{xy}\,dx\,dz\right)}_{\text{avg force}} \times \underbrace{\left(\gamma_{xy}\,dy\right)}_{\text{disp}}$$

Therefore, the strain energy density in this case of only shear stress τ_{xy} is:

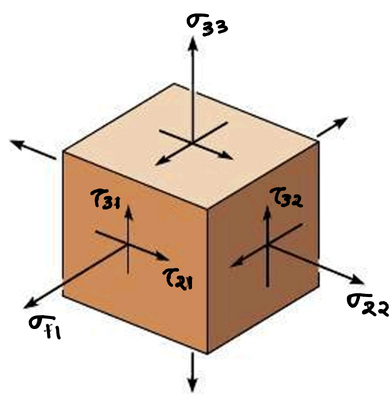
$$\bar{u} = \frac{1}{2}\,\tau_{xy}\,\gamma_{xy}$$

Since for a linear elastic isotropic material, $\tau_{xy} = G\,\gamma_{xy}$, we can write the strain energy density using two alternative defns:

$$\bar{u} = \frac{1}{2}\,\frac{\tau_{xy}^2}{G} = \frac{1}{2}\,G\,\gamma_{xy}^2$$

For general state of stress, the strain energy density becomes:

$$\bar{u} = \frac{1}{2}\Big(\sigma_{11}\,\epsilon_{11} + \sigma_{22}\,\epsilon_{22} + \sigma_{33}\,\epsilon_{33} + \tau_{21}\,\gamma_{21} + \tau_{31}\,\gamma_{31} + \tau_{32}\,\gamma_{32}\Big)$$



For linear elastic isotropic material, the stress-strain relation is given by 3D Hooke's law of elasticity:

$$\epsilon_{11} = \frac{1}{E}\left[\sigma_{11} - \nu\left(\sigma_{22} + \sigma_{33}\right)\right], \qquad \gamma_{12} = \frac{\tau_{12}}{G}, \;\ldots$$

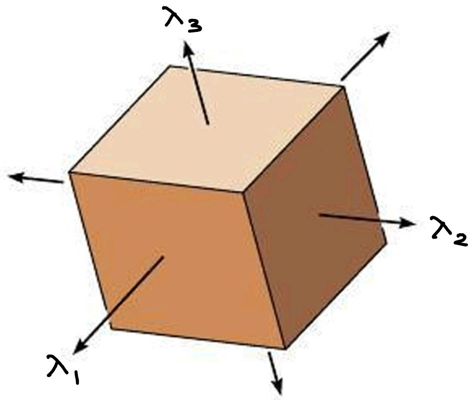
the strain energy density is obtained as:

$$\bar{u} = \frac{1}{2E}\left(\sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2\right) - \frac{\nu}{E}\left(\sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11}\right)$$

$$+ \frac{1}{2G}\left(\tau_{12}^2 + \tau_{13}^2 + \tau_{23}^2\right)$$

If one considers strain energy density in the principal coordinates (recall that for linear elastic isotropic materials, principal directions for stress and strain coincide), it reduces to a simpler form:

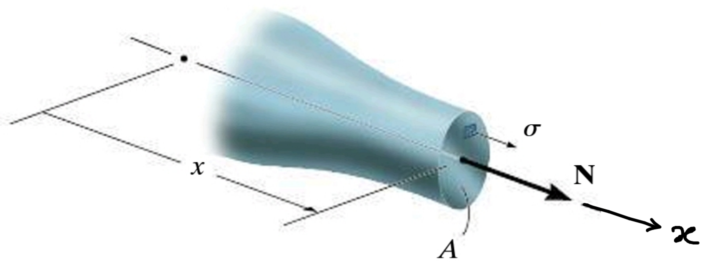
$$\bar{u} = \frac{1}{2E}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2\right) - \frac{\nu}{2E}\left(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1\right)$$



Elastic strain energy for various types of loading

In order to apply work-energy methods to solve problems, we must obtain expressions for elastic strain energy stored in a body when a member is subjected to an axial load, torsional loads, and bending loads.

Axial load



Stress, $\sigma_x = \dfrac{N(x)}{A(x)}$ Internal force on the c/s at x

Strain, $\epsilon_x = \dfrac{d\,u_x(x)}{dx}$ disp. of the c/s at x

$$dV = A(x)\,dx$$

Strain energy,

$$U_i = \int_V \frac{\sigma_x^2}{2E}\,dV = \frac{1}{2}\int_0^L \left(\frac{N(x)}{A(x)}\right)^2 \frac{1}{E}\,A(x)\,dx = \frac{1}{2}\int_0^L \frac{N^2(x)}{E(x)A(x)}\,dx$$

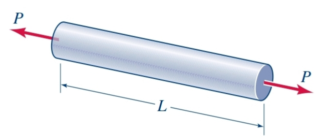
$$U_i = \int_V \frac{E\,\epsilon_x^2}{2}\,dV = \frac{1}{2}\int_0^L E(x)\left(\frac{du_x(x)}{dx}\right)^2 A(x)\,dx = \frac{1}{2}\int_0^L E(x)\,A(x)\left(\frac{du}{dx}\right)^2 dx$$

Therefore, elastic strain energy due to axial deformation can be expressed as:

$$U_i = \frac{1}{2}\int_0^L \frac{N^2(x)}{A(x)\,E(x)}\,dx \quad \text{and} \quad U_i = \int_0^L E(x)\,A(x)\left(\frac{du(x)}{dx}\right)^2 dx$$

For a uniform linear elastic rod subjected to axial end loads,

$$\boxed{U_i = \frac{P^2L}{2AE}}$$



Torsional load (for circular c/s members)

We will consider the stored internal strain energy of CIRCULAR rods. Recall that, shear strain γ in torsion is given by

$$\gamma(x, r) = r \frac{d\phi(x)}{dx}$$

angle of twist of the c/s at x

and the shear stress τ is related to the internal torque at a c/s by the equation

$$\tau(x, r) = \frac{T(x)\, r}{J(x)}$$

We obtain the following expression for strain energy due to torsion, expressed in terms of the internal torque T:

$$U_i = \frac{1}{2} \int \frac{1}{G(x)} \left[\frac{T(x)\, r}{J(x)} \right]^2 dV$$

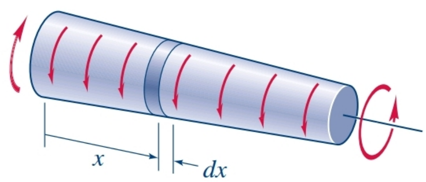
$$= \frac{1}{2} \int_0^L \int_A \frac{1}{G(x)} \left[\frac{T^2(x)}{J^2(x)} \right] r^2 dA \, dx$$

$$= \frac{1}{2} \int_0^L \frac{1}{G(x)} \frac{T^2(x)}{J^{\cancel{2}}(x)} \cancel{J(x)} \, dx \qquad \int_A r^2 dA \equiv J(x)$$

$$= \frac{1}{2} \int_0^L \frac{T^2(x)}{G(x) J(x)} dx$$

Alternative expression in terms of $\frac{d\phi}{dx}$

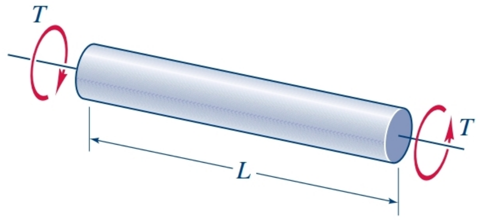
$$U_i = \frac{1}{2} \int_0^L G(x) J(x) \left(\frac{d\phi}{dx} \right)^2 dx$$



$G \equiv G(x)$

For a uniform circular linear elastic rod subjected to end torque T, as shown, the elastic strain energy is:

$$\boxed{U_i = \frac{T^2 L}{2GJ}}$$



Bending of beams — Bending strain energy

A beam has bending stress σ_{xx} and shear stress τ_{xy} that vary with position on the beam. We have seen from the formula of strain energy density for general state of stress that the contributions of normal stress and shear stress to strain energy can be treated separately.

We begin by considering the bending stress σ_{xx} (and strain ϵ_{xx}) and we employ Euler-Bernoulli beam theory.

The bending strain in a fibre present at a distance y from NA is related to the transverse deflection curve and to the deflection $v(x)$

$$\epsilon_{xx} = -\frac{y}{R(x)} \approx -y\frac{d^2 v}{dx^2}$$

transverse disp.

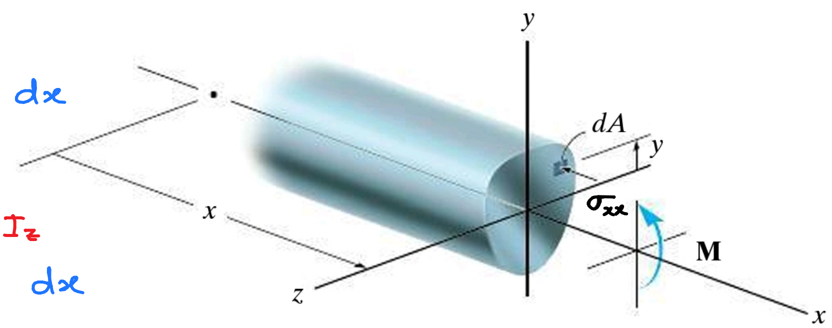
radius of curvature

Assuming Young's modulus may be varying, $E(x)$, the bending stress is:

$$\sigma_{xx} = -\frac{M(x)\, y}{I_z(x)}$$

Using the relation of bending stress

we get:



$$U_i = \frac{1}{2}\int_0^L \int_A \frac{1}{E(x)}\left[-\frac{M(x)\,y}{I_z(x)}\right]^2 dA\,dx$$

$$= \frac{1}{2}\int_0^L \frac{M^2(x)}{E(x)\,I_z^{\cancel{2}}(x)} \cancelto{I_z}{\int_A y^2\,dA}\; dx$$

$$= \frac{1}{2}\int_0^L \frac{M^2(x)\,dx}{E(x)\,I_z(x)}$$

In terms of the curvature $\frac{d^2v}{dx^2}$, the strain energy due to bending is given by

$$U_i = \frac{1}{2}\int_0^L \int_A E(x)\left[\frac{d^2v}{dx^2}\right]^2 y^2\,dA\,dx$$

$$= \frac{1}{2}\int_0^L E(x)\,I(x)\left(\frac{d^2v}{dx^2}\right)^2 dx$$

Bending of beams — Shear strain energy

The shear stress in a beam also contributes to the strain energy that is stored in the beam. The shear-stress formula is:

$$\tau_{xy} = \frac{V(x)\,Q(y)}{I_z(x)\,b(y)}$$

transverse shear force

The strain energy is:

$$U_i = \frac{1}{2}\int_0^L \int_A \frac{1}{G(x)}\left[\frac{V(x)\,Q(y)}{I_z(x)\,b(y)}\right]^2 dA\,dy$$

$$U_i = \frac{1}{2} \int_0^L \left[\frac{V^2(x)}{G(x)\, I^2(x)} \int \frac{Q^2(x,y)}{b^2(y)}\, dA \right] dx$$

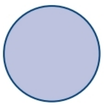
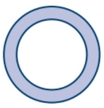
To simplify this expression for U_i, we define a new cross-sectional property f_s called the form factor for shear. Let

$$f_s = \frac{A(x)}{I_z^2(x)} \int_A \frac{Q^2(x,y)}{b^2(y)}\, dA$$

Strain energy due to shear in bending:

$$U_i = \frac{1}{2} \int_0^L \frac{f_s V^2 dx}{G A} = \frac{1}{2} \int_0^L \frac{V^2 dx}{k G A}$$

shear correction factor

	Section	f_s
	Rectangle	$\frac{6}{5}$
	Circle	$\frac{10}{9}$
	Thin tube	2
	I-section or box section	$\approx \frac{A}{A_{\text{web}}}$