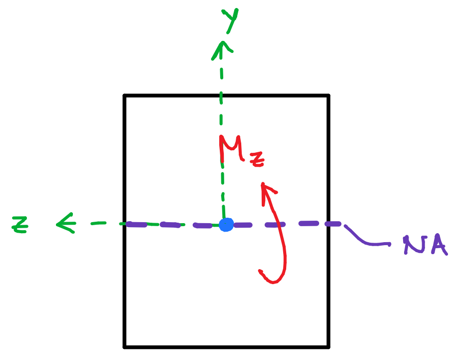
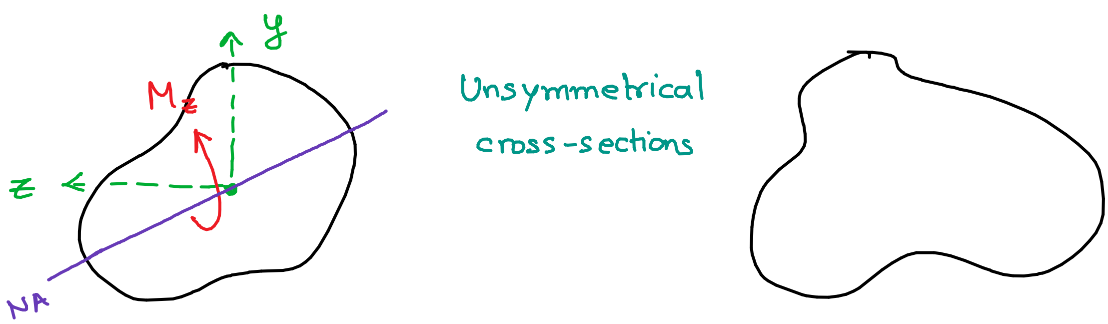
Uniform bending of unsymmetrical beams

Till now, we have discussed about symmetrical beam bending. For example, we have a rectangular beam C/s with the y- and z-axis with their origin at the centroid of the C/s. We have a moment applied about the z-axis, M_z. For the analysis, we considered the moment to be applied such that the Neutral Axis coincides with the axis about which the moment is applied (here the z-axis)



$\Rightarrow$ Neutral axis coincides with the axis of applied moment (in this case z-axis)

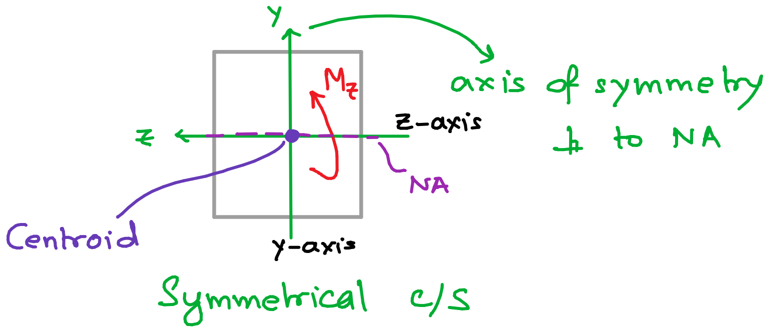
However, in general, the neutral axis may not coincide with the axis of applied moment. For an arbitrary cross-section such as these,

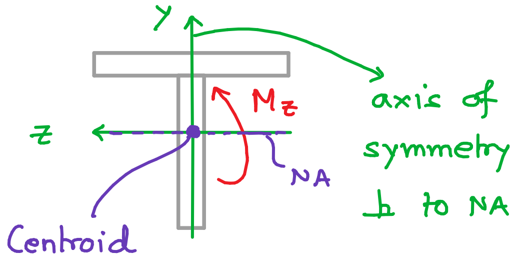


if you apply moment M_z about the z-axis, the neutral axis would be inclined, which means you are applying moment about one axis but you are getting bending about a different axis.

Symmetrical bending occurs when the beam

- C/s has an axis of symmetry perpendicular to the neutral axis

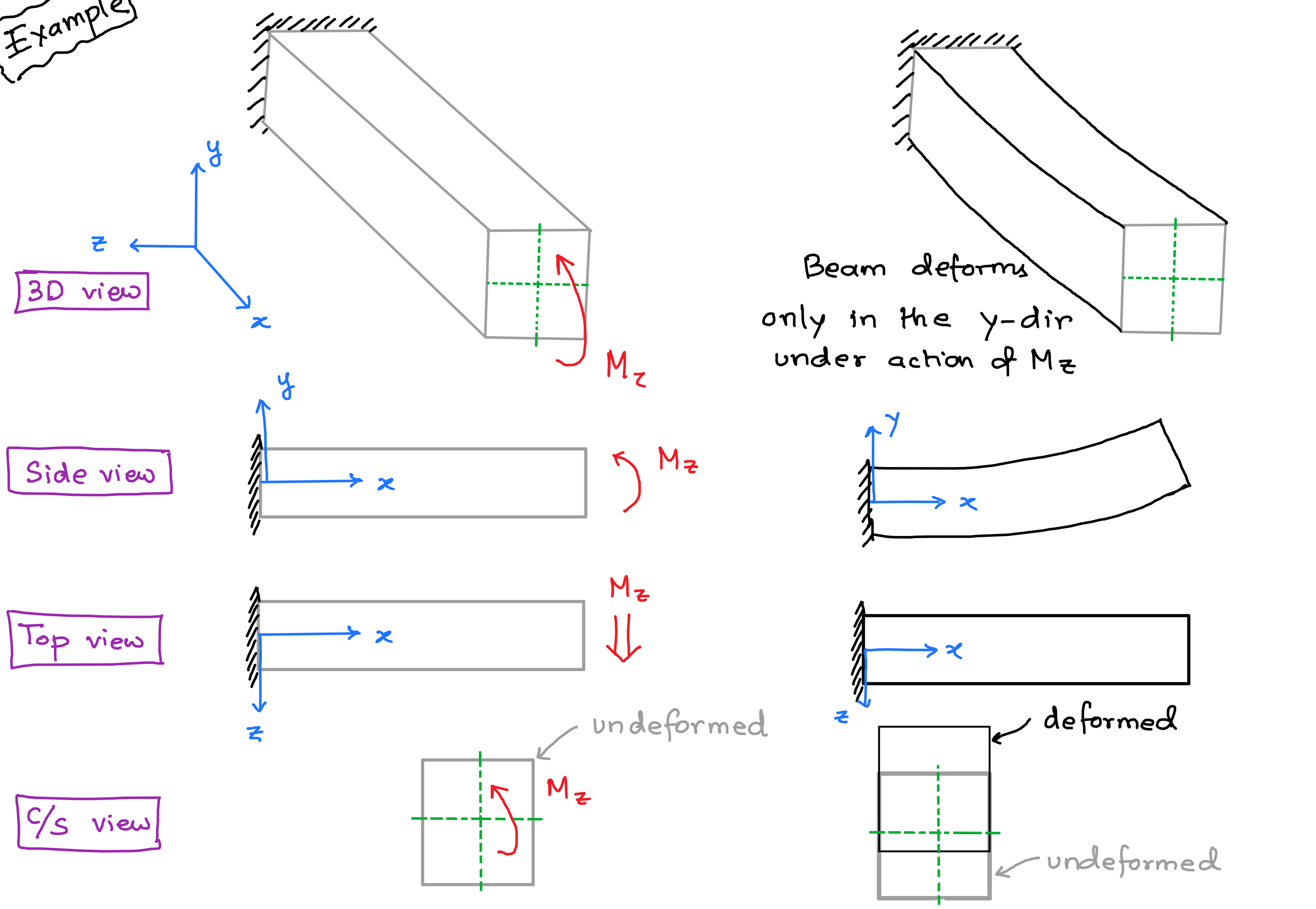




- is subjected to a moment (or a transverse load) about an axis of symmetries of the cross-section.

In such cases, the neutral axis coincides with an axis of symmetry leading to symmetrical bending

Example

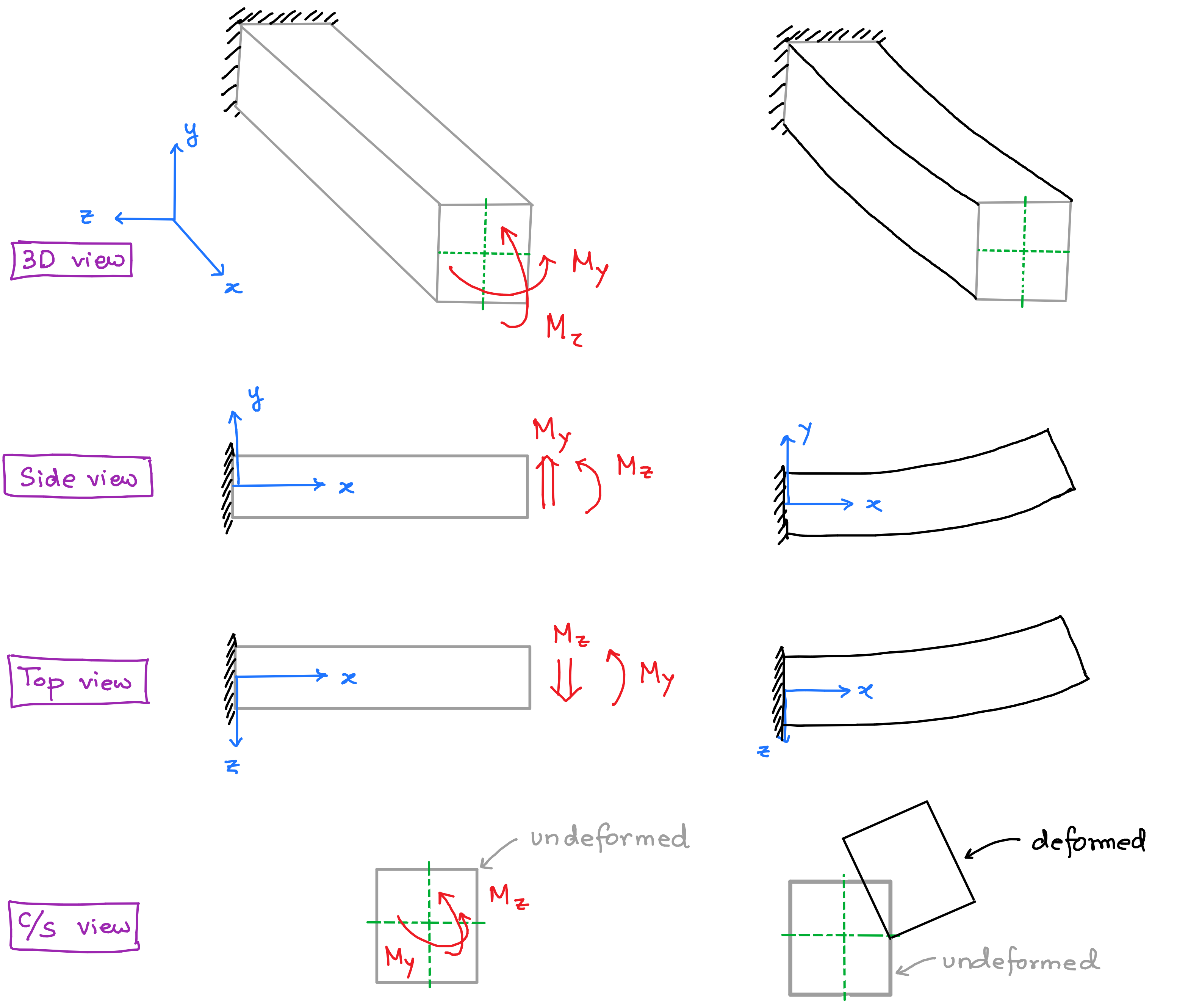


Unsymmetrical bending is where a beam undergoes displacements in both y- and z-directions. It is caused by

- Beam carrying moments (or transverse loads) in both y- and z-directions, OR/AND

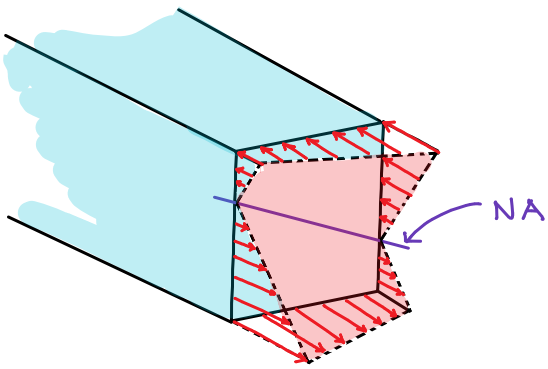
- The cross-section is not symmetrical

The result is that the neutral axis does not coincide with the axes of symmetry.



Just like before, we first consider pure bending of unsymmetriccal beams and derive the bending stress distribution over a cross-section under pure bending. We shall see that our results will apply to both

- beams of symmetrical C/s but loaded unsymmetrically,
- beams of unsymmetrical C/s



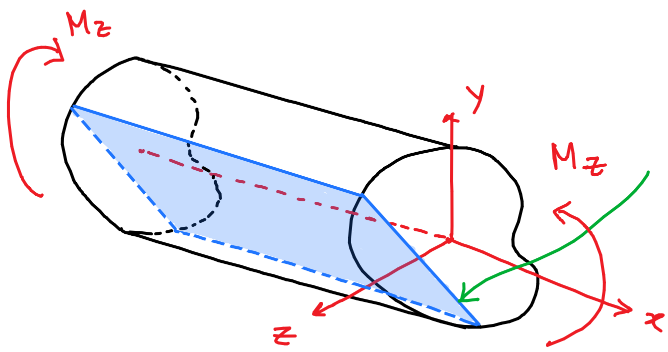
Symmetric beam loaded unsymmetrically

Our approach to deriving the bending stress distribution will be very similar to that followed earlier. We still assume that σ_{yy} and σ_{zz} are negligible compared to σ_{xx}. Due to pure bending, internal resistive BM would be constant and shear force would be zero. So we are going to use some of the results already obtained for pure bending of symmetrical beams.

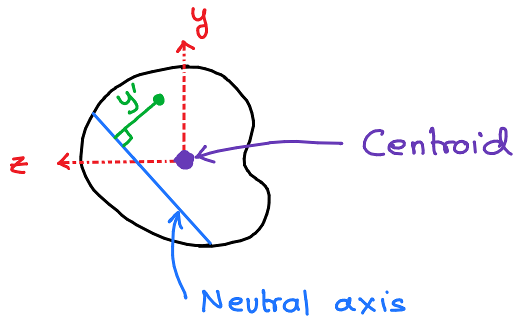
It can be shown that in unsymmetrical bending, the NA still passes through the centroid of the C/s, however, the NA will not be aligned with the z-axis. (See next page for a proof)

Neutral axis still pass through centroid but with unknown alignment

Consider an unsymmetrical beam C/s under pure bending



Let us assume that this is the neutral plane



C/s view

Assuming the beam bends into an arc of a perfect circle of radius R, the bending strain ϵ_{xx} would be:

$$\epsilon_{xx} = -\frac{y'}{R}$$

(following the same logic from symmetric beam under pure bending)

Therefore, $\sigma_{xx} = E\epsilon_{xx} = -E\frac{y'}{R}$

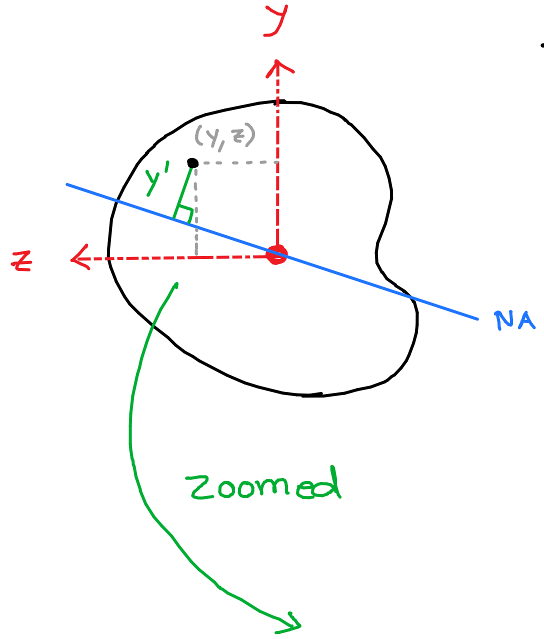
Since there is no axial force in the C/s, thus $\overset{+}{\rightarrow}\Sigma F_x = 0$

$$\overset{+}{\rightarrow}\Sigma F_x = 0 \quad \Rightarrow \quad \iint_{A_0} \sigma_{xx}\, dA = -\frac{E}{R}\underbrace{\iint_{A_0} y'\, dA}_{} = 0$$

↳ So the NA must pass through the centroid of the C/s even for unsymmetrical beam C/s.

Bending stress determination

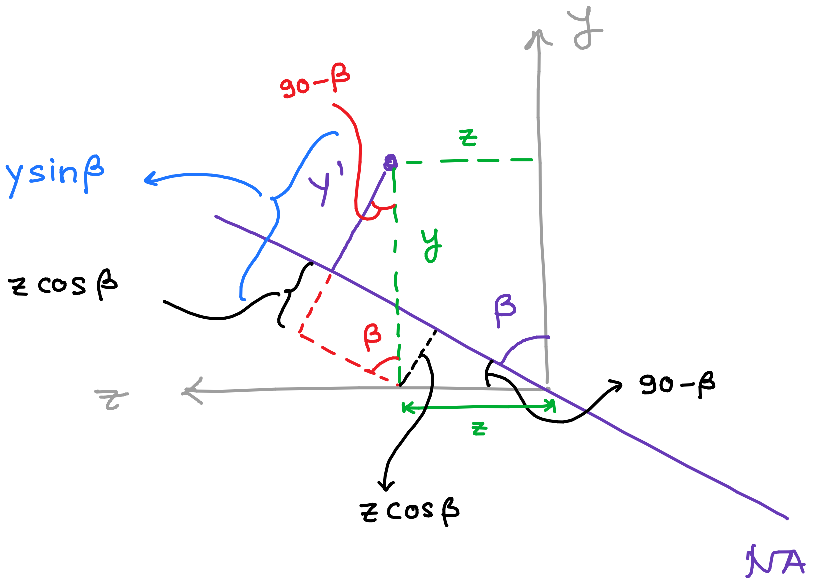
We will now draw the NA s.t it passes through the centroid



Think of a pt A (y, z) in the c/s at a distance y' from the NA

We want to describe y' in terms of (y, z)

So we have to do some geometry here.



So based on the geometry,

$$y' = y\sin\beta - z\cos\beta$$

Therefore,

$$\epsilon_{xx} = -\frac{y'}{R} = -\frac{(y\sin\beta - z\cos\beta)}{R}$$

bending stress $\longrightarrow$ $\sigma_{xx} = -E\,\epsilon_{xx} = -\dfrac{E}{R}\left(y\sin\beta - z\cos\beta\right)$

Now lets obtain the moment about the C/s's centroid due to the bending stress σ_{xx} which should be equal to externally applied moment vector $\vec{M}_0 = M_y \hat{j} + M_z \hat{k}$

$$\vec{M}_{\text{abt centroid } O} = \iint_{A_0} (y\hat{j} + z\hat{k}) \times (\sigma_{xx}\hat{i})\, dA$$

$$= \iint_{A_0} (y\hat{j} + z\hat{k}) \times \frac{E}{R}(z\cos\beta - y\sin\beta)\hat{i}\, dA$$

$$= \frac{E}{R}\iint_{A_0} \left[(y^2\sin\beta - yz\cos\beta)\hat{k} + (z^2\cos\beta - yz\sin\beta)\hat{j}\right] dA$$

We define the following second moments of inertia of C/s area:

$$\underbrace{\iint_{A_0} y^2 dA = I_{zz}}_{\text{moment of inertia about } z\text{-axis}}, \quad \underbrace{\iint_{A_0} z^2 dA = I_{yy}}_{\text{moment of inertia about the } y\text{-axis}}, \quad \underbrace{\iint_{A_0} yz\, dA = I_{yz}}_{\text{cross moment of inertia}}$$

With these quantities, we get:

$$\vec{M}_0 = \frac{E}{R}\left[\underbrace{(I_{zz}\sin\beta - I_{yz}\cos\beta)}_{M_z}\hat{k} + \underbrace{(I_{yy}\cos\beta - I_{yz}\sin\beta)}_{M_y}\hat{j}\right]$$

$$\Rightarrow \quad \left.\begin{aligned} M_y &= (I_{yy}\cos\beta - I_{yz}\sin\beta)\frac{E}{R} \\ M_z &= (I_{zz}\sin\beta - I_{yz}\cos\beta)\frac{E}{R} \end{aligned}\right\}$$ Solve these two equations to obtain β & R!

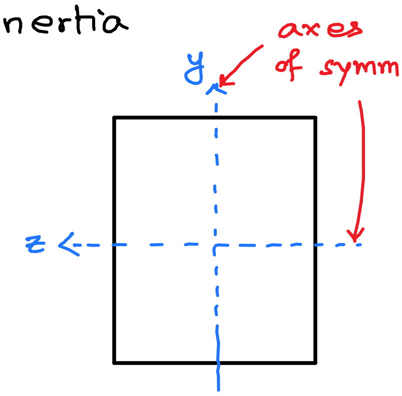
$$\Rightarrow \quad \boxed{\frac{M_y}{M_z} = \frac{I_{yy}\cos\beta - I_{yz}\sin\beta}{I_{zz}\sin\beta - I_{yz}\cos\beta}}$$ $\longrightarrow$ Obtain β from here with known M_y and M_z

Suppose $M_y = 0 \Rightarrow I_{yy} \cos\beta - I_{yz} \sin\beta = 0$

$$\Rightarrow \boxed{\tan\beta = \frac{I_{yy}}{I_{yz}}}$$

$\longrightarrow$ you can get the inclination β of the Neutral axis

In the last two lectures, we encountered symmetrical bending with rectangular c/s, which has two axes of symmetry along the y- and z-axis. As such, the principal axes of inertia for the c/s area coincide with y- and z-axes. Therefore, the cross moment of inertia of area will vanish for a rectangular c/s.



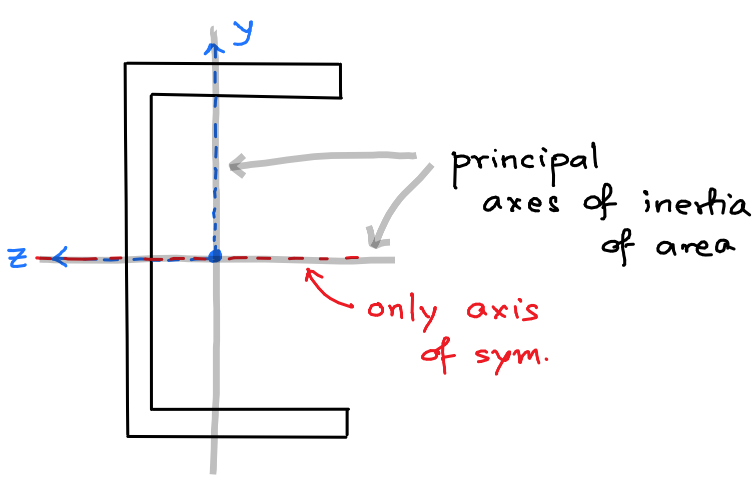
$I_{yz} = 0$

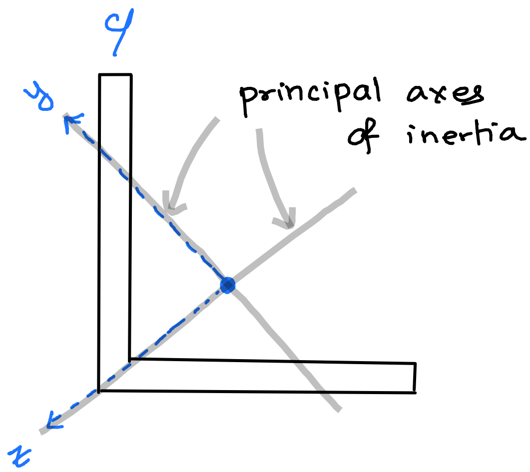
($\because$ y and z axes coincide with principal axes)

Remember

- If the c/s area has an axis of symmetry, the principal axes will always be oriented along the axis of symmetry and perpendicular to it

- Cross moment of inertia of a c/s area is zero provided y and z axes coincide with the principal axes of inertia for c/s

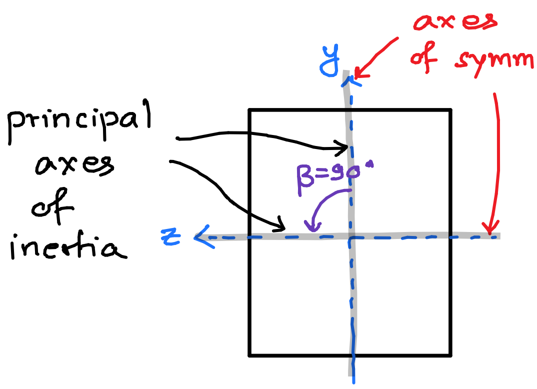




For a rectangular C/S,

$$\tan\beta = \frac{I_{yy}}{\cancel{I_{yz}}^{\,\to 0}} = \infty$$

$$\Rightarrow \quad \beta = 90°$$



$I_{yz} = 0$
($\because$ y and z axes coincide with principal axes)

$\therefore$ the neutral axis (the axis of bending) coincides with the z-axis. This is a result we have already seen before. Only that now we have got a more general formula

$$\boxed{\frac{M_y}{M_z} = \frac{I_{yy}\cos\beta - I_{yz}\sin\beta}{I_{zz}\sin\beta - I_{yz}\cos\beta}}$$

Once β is determined, we can determine the radius of curvature from the relation of M_y (or M_z)

$$R = \frac{E\left(I_{yy}\cos\beta - I_{yz}\sin\beta\right)}{M_y}$$

Then, the bending stress becomes :

$$\sigma_{xx} = -\frac{E\,y'}{R} = -\frac{M_y\left(y\sin\beta - z\cos\beta\right)}{\left(I_{yy}\cos\beta - I_{yz}\sin\beta\right)}$$

If you also plugging the value of β in this formula :

$$\sigma_{xx} = \frac{M_z\left(y\,I_{yy} - z\,I_{zz}\right) + M_y\left(y\,I_{yz} - z\,I_{zz}\right)}{I_{yz}^2 - I_{yy}\,I_{zz}}$$

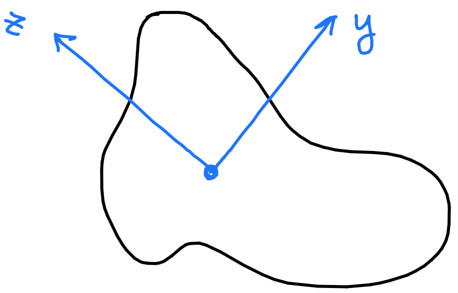
This is a formula for finding bending stress for a generalized beam C/s, applicable for symmetrical as well as unsymmetrical beams

$$\sigma_{xx} = \frac{M_z (y I_{yy} - z I_{zz}) + M_y (y I_{yz} - z I_{zz})}{I_{yz}^2 - I_{yy} I_{zz}}$$

Comparing the above relation for symmetrical beam bending stress with only M_z present

$$\sigma_{xx} = \frac{M_z y}{I_{zz}}$$

Now let us suppose that the 'y' and 'z' axes of the C/s are oriented along the principal axes of the C/s



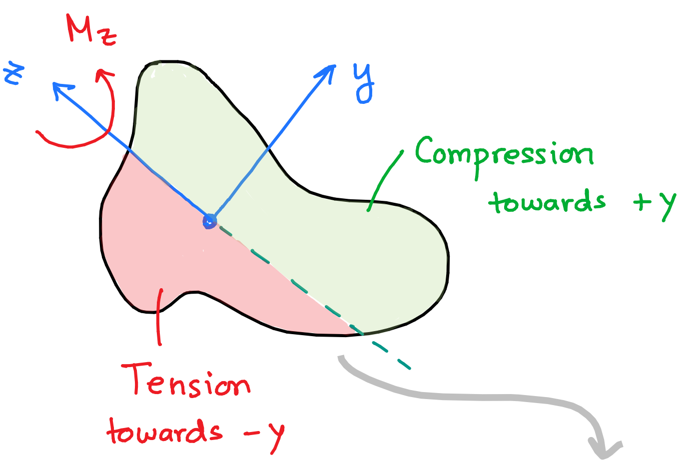
$\Rightarrow \quad I_{yz} = 0$!

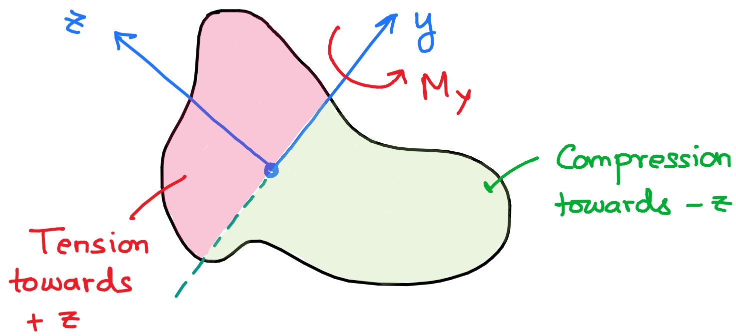
Plug $I_{yz} = 0$ in the general formula, and we get:

$$\boxed{\sigma_{xx} = - \frac{M_z y}{I_{zz}} + \frac{M_y z}{I_{yy}}}$$

Why is there a negative sign for $\frac{M_z y}{I_{zz}}$?

If $M_y = 0$, then $\sigma_{xx} = \frac{-M_z y}{I_{zz}}$





Since we consider σ_{xx} is +ve under tension, therefore the −ve sign