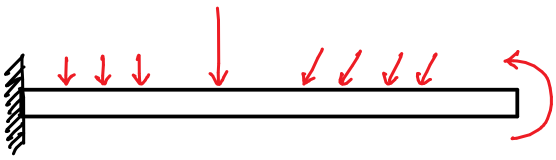
Bending of beams

When a slender member is subjected to transverse loading we say it is a beam.

loading perpendicular to beam axis



c/s

In general, beams are long, straight bars having a constant c/s area. Often they are classified as to how they are supported.

Simply supported beam ↔ pinned at one end, roller at other end

Simply supported beam

Cantilever beam ↔ fixed at one end, free at other end

Cantilevered beam

Overhanging beam ↔ one end free / fixed, extended over a roller support

Overhanging beam

Beams are important structural elements. They are used to support the floor of a building, deck of a bridge, wing of an aircraft, the boom of a crane, bones in our bodies, etc.

Shear force and Bending moment diagrams

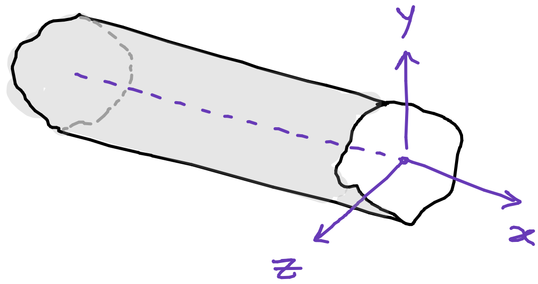
Because of the applied transverse loads, beams develop an internal resistive shear force and bending moment. These internal shear force and bending moment can vary from point to point along the length (or the axis) of the beam.

The internal shear force and bending moment functions can be plotted and represented by graphs called shear force and bending moment diagrams, SFD & BMD.

The maximum values of shear force, V and bending moment, M can be obtained from these graphs.

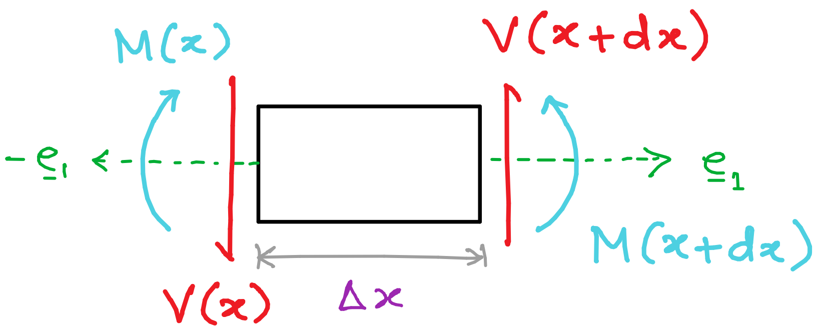
Procedure for drawing SFD & BMD

- Determine support reactions
- For beam problems, x-axis (or $\underline{e}_1$-dir) is considered along the beam length



Beam length is along x-axis

- Usually, we specify an origin of coordinate system at the left end
- Take sections of the beam at different values of x and draw free-body-diagrams.
- Assert a sign convention for SF and BM



Positive sign convention for SF & BM

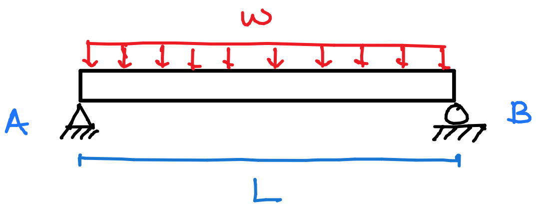
- When a force or moment acts on a positive face in a positive coordinate direction, the corresponding force/moment is positive
- When a force or moment acts on a negative face in a negative coordinate direction, the corresponding force/moment is also positive

- The shear forces are obtained by summing forces perpendicular to the beam axis

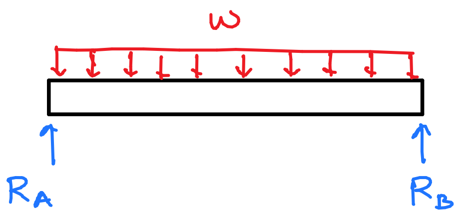
- The bending moment is obtained by summing moments abt the sectioned end of the segment

- Plot the SFD and BMD. If numerical values of V and M are positive, the values are plotted above the x-axis, whereas negative values are plotted below the axis.

Ex:



FBD

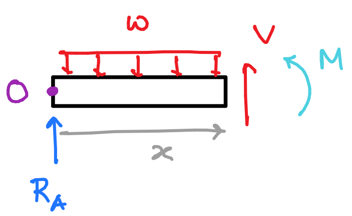


- Support reactions

$$R_A = R_B = \frac{wL}{2}$$

SFD and BMD

Consider origin at the left end and take a segment of beam



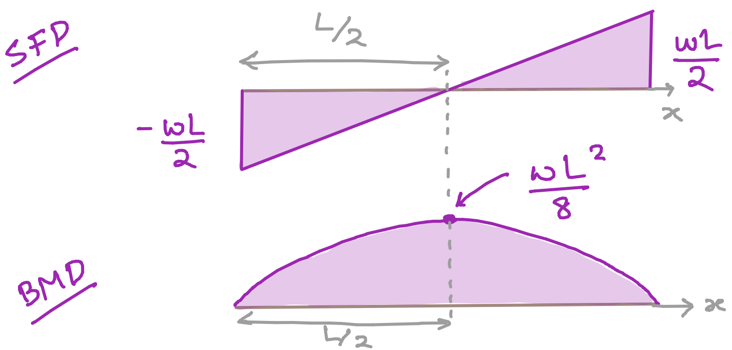
$$+\uparrow \sum F_y = 0 \Rightarrow \overset{\frac{wL}{2}}{\cancel{R_A}} + V = wx$$

$$\Rightarrow V = -w\left(\frac{L}{2} - x\right)$$

$$\circlearrowleft + \sum M_O = 0 \Rightarrow Vx - wx\left(\frac{x}{2}\right) + M = 0$$

$$\Rightarrow M = w\frac{L}{2}x - wx^2 + \frac{wx^2}{2}$$

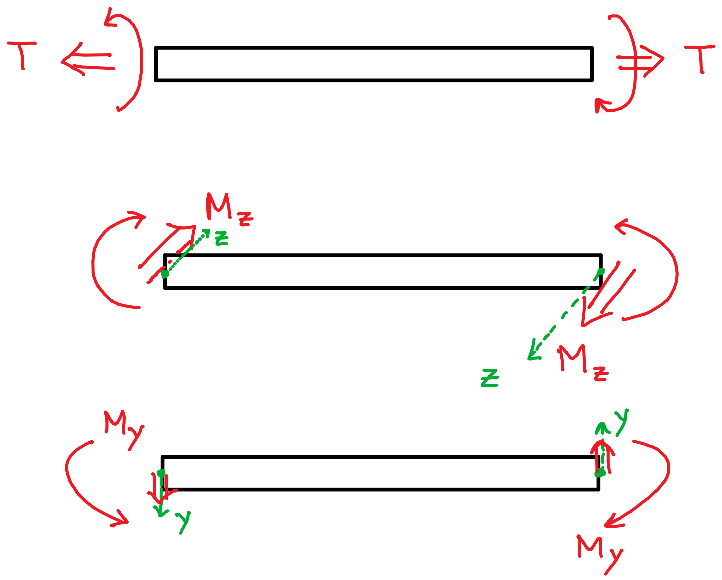
$$= \frac{w}{2}\left(Lx - x^2\right)$$



We have now learned how to draw shear force and bending moment diagrams for beams subjected to transverse loadings.

But while drawing them, we assumed the beams to not deform.

But the beams curve $\Rightarrow$ That is called BENDING

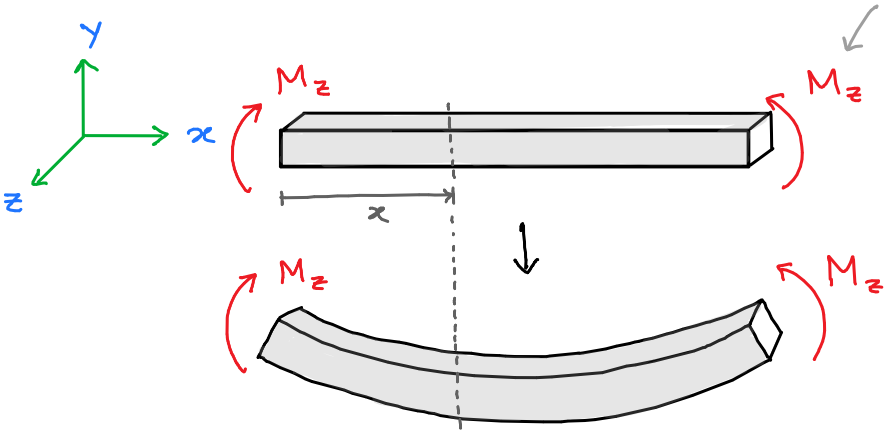


Torque moment vector acts along the length of the beam $\xrightarrow{\text{causing}}$ twist

Bending moment acts $\perp$ to the axis of the beam $\xrightarrow{\text{causing}}$ bending

Now we will take another step and learn how beams deform or get curved when they are subjected to such loads. The simplest case to begin with is that what we call as PURE BENDING of beams, where a constant moment acts on the beam

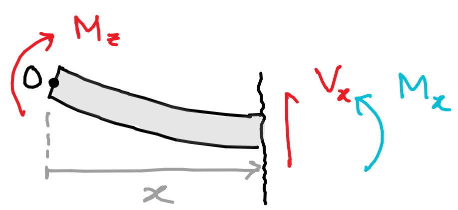
applied moment about z-axis



Straight before deformation

Curved after deformation

We find that for this type of loading (i.e. constant moment applied to the beam), there is no shear force arising in the beam, and the beam is only subjected to a constant bending moment along its length. We can verify these by doing force and moment balance at an arbitrary section at a distance x



$V_x \rightarrow V$ at a dist x

$M_x \rightarrow M$ at a dist x

$$+\uparrow \sum F_y = 0$$

$$\Rightarrow \quad V_x = 0$$

No internal shear force

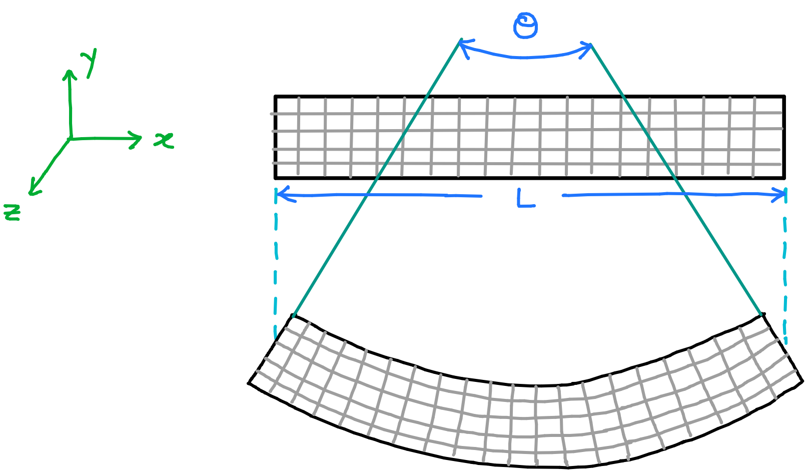
$$\circlearrowleft + \sum M_o = 0$$

$$\Rightarrow \quad M_x + \cancelto{0}{V_x}\, x - M_z = 0$$

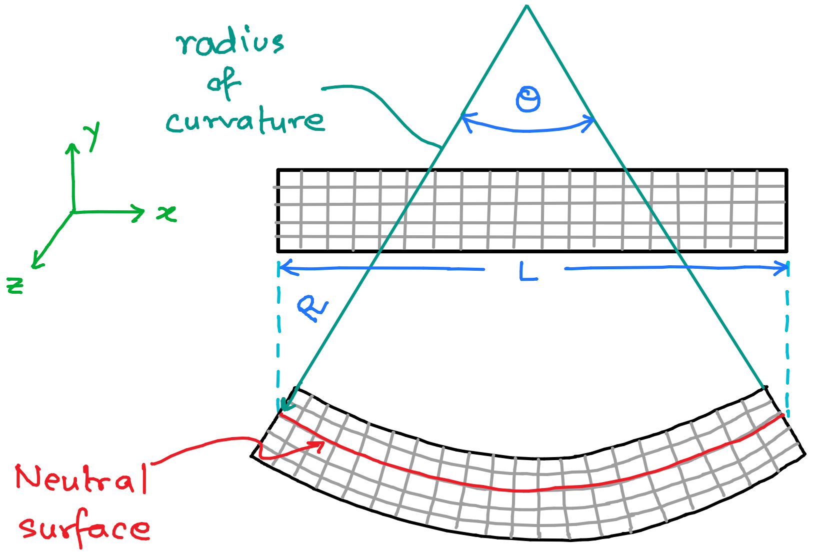
$$\Rightarrow \quad M_x = M_z$$

Bending moment is constant everywhere

For such a case where there is no shear force and a constant bending moment throughout the beam with constant cross-section, the beam bends in such a way that it represents an arc of a perfect circle. As such, all lines (or fibers) of the beam along its length deform to become arcs of a circle



- The longitudinal lines become curved into arcs while the vertical transverse lines remain straight, yet undergo a rotation

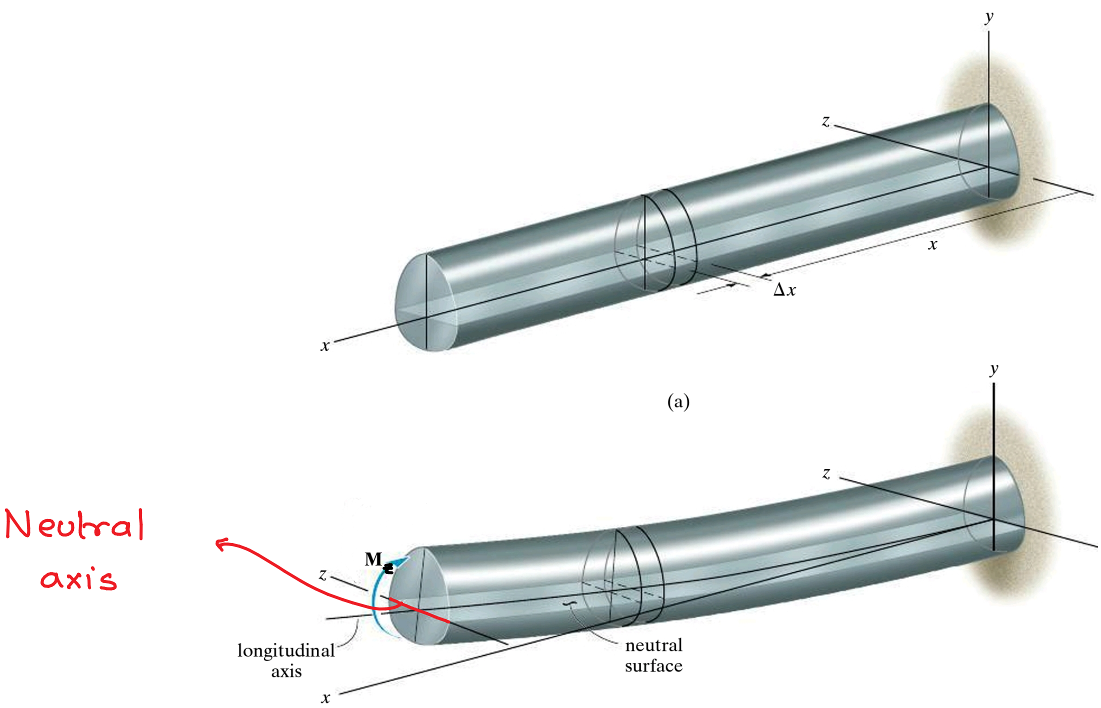


Assumption: Planar C/s remains plane after deformation

- The longitudinal lines become curved into arcs while the vertical transverse lines remain straight, yet undergo a rotation

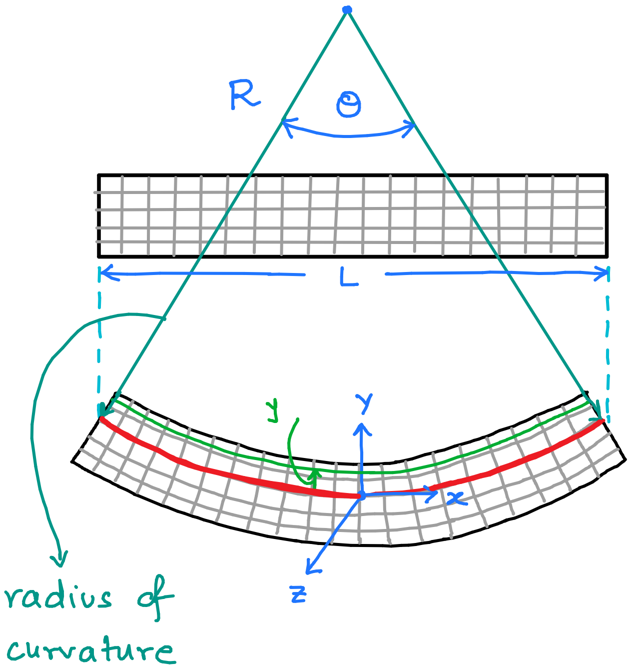
- The bending moment causes the fibers within the top portion to shorten/compress and those in the bottom portion to stretch

- Therefore, between the two regions of fibers getting compressed and elongated, there must be a surface whose fibers do not undergo any elongation or compression. This surface is called the NEUTRAL PLANE as all longitudinal lines on this plane do not undergo any change in length. In particular, the z-axis, lying in the plane of the cross-section and about which the C/s rotates is called the neutral axis



Longitudinal / bending strain

Let us now find the amount of elongation/compression of the longitudinal line elements which are parallel to the beam's axis. The neutral line is shown in red. Note that we donot know yet the location of this neutral line exactly. We will consider the x-axis to lie in the neutral surface.



Consider a longitudinal fiber (shown in green line) at a distance y from the neutral line.

If the radius of the neutral line is R, the radius of the circle corresponding to the green line will be $(R-y)$. If the angle subtended by these arcs at the center is θ, then the length of the neutral line will be

$$l_n = R\theta$$

This length of neutral line must be equal to the undeformed length of the beam, L.

$$l_n = R\theta = L$$

Similarly, the length of green line would be

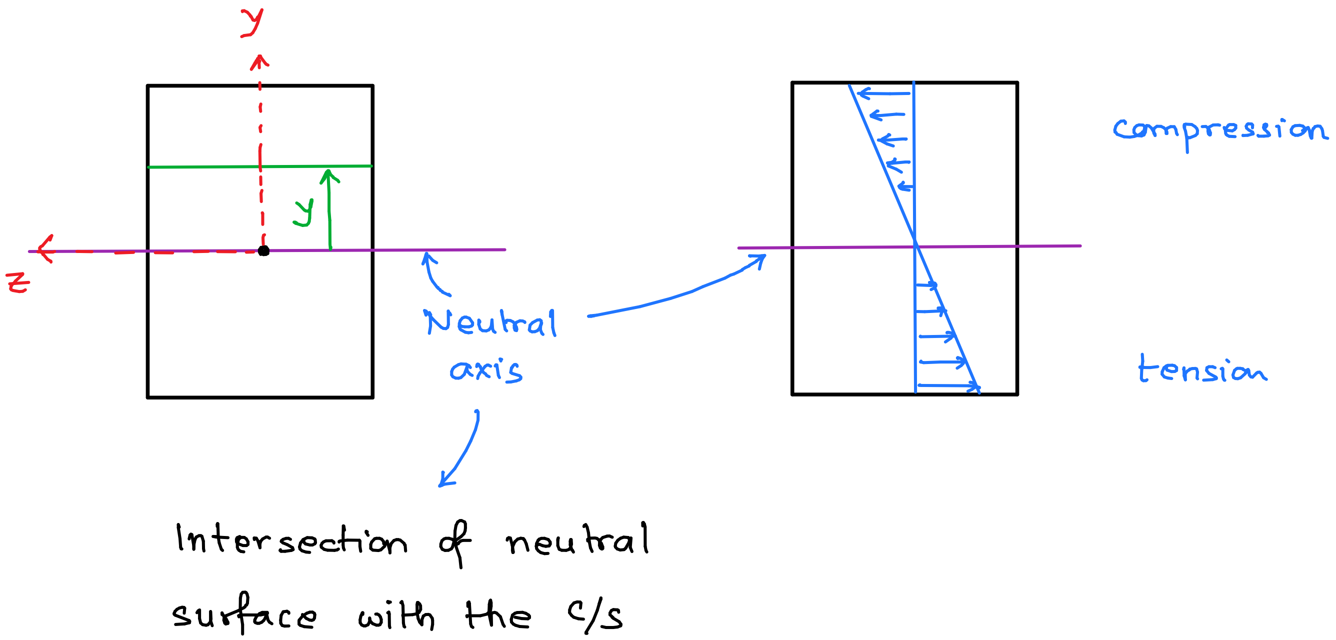
$$l_g = (R-y)\,\theta = (R-y)\,\frac{R\theta}{R} = (R-y)\,\frac{L}{R} = \left(1-\frac{y}{R}\right)L$$

As the undeformed length of all longitudinal lines is L, the longitudinal strain of the green line will be

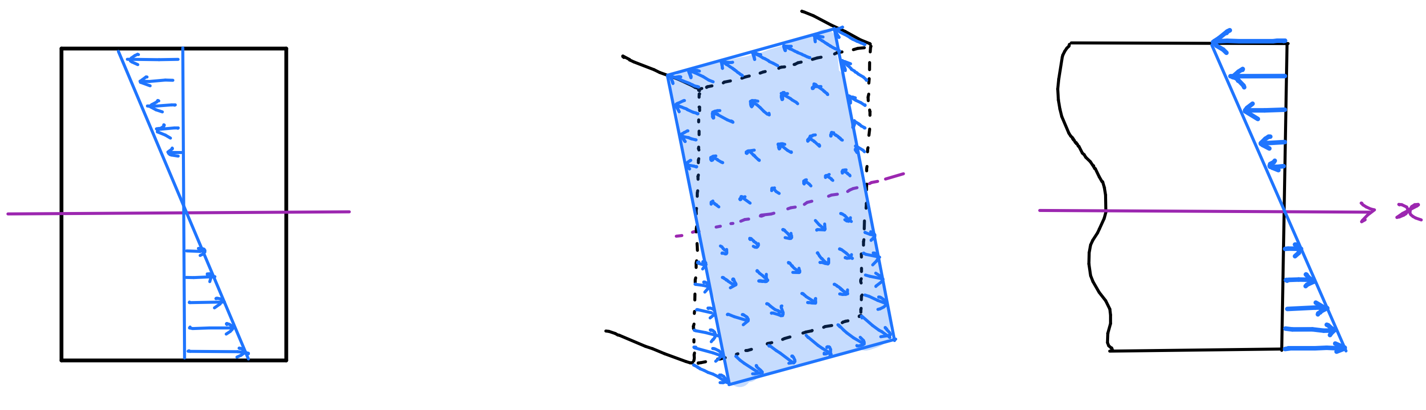
$$\epsilon_b = \epsilon_{xx} = \frac{\Delta L}{L} = \frac{l_g - L}{L} = -\frac{y}{R}$$

bending strain

We have thus obtained longitudinal strain along x-axis (hence ϵ_{xx}) for a general longitudinal line at a distance y from the neutral line.



We see that any longitudinal line with a constant y (like the green line) will have the same ϵ_{xx}.

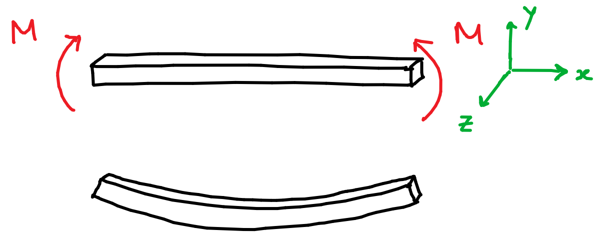


Representation in 2D

Actual in 3D

Obtain the bending stress

For the case of pure bending, we only apply moments at the two ends of the beam.



We donot apply any external force on the front, back, top or bottom surfaces of the beam. On these surfaces, the traction components will be zero.

However, we make the assumption that σ_{yy} and σ_{zz} will be zero at all points within the beam. This is a reasonable approximation because as the beam bends, it is allowed to relax freely in the y- and z-directions. Hence, only σ_{xx} is non-zero within the beam.

To get the bending stress, we use the following stress-strain relationship for isotropic materials:

$$\epsilon_{xx} = \frac{1}{E}\left(\sigma_{xx} - \nu(\cancelto{0}{\sigma_{yy}} + \cancelto{0}{\sigma_{zz}})\right)$$

We intentionally use this relationship since we know σ_{yy} and σ_{zz} are zero, however, ϵ_{yy} & ϵ_{zz} may not be zero.

$$\Rightarrow \quad \sigma_{xx} = E\,\epsilon_{xx}$$

$$= -E\,\frac{y}{R}$$

Notice that σ_{xx} is compressive above the neutral axis and tensile below it.

but where is the neutral axis located?

Location of neutral axis

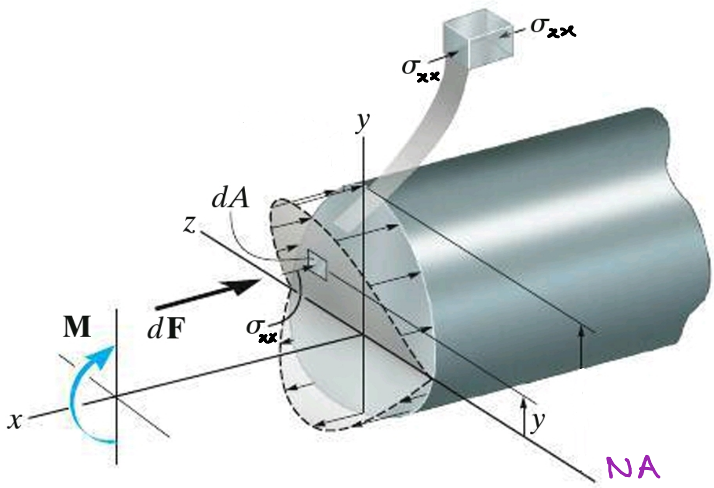
In a pure bending scenario, every cross-section only has bending moment acting on it. There is no external force acting in the cross-section. Hence, the integration of σ_{xx} in the C/s plane must be equal to zero;

$$\int_{A_0} \sigma_{xx} \, dA = 0$$

$$\Rightarrow -\int_{A_0} \frac{Ey}{R} \, dA = 0$$

non-zero terms

$$\Rightarrow \int_{A_0} y \, dA = 0$$

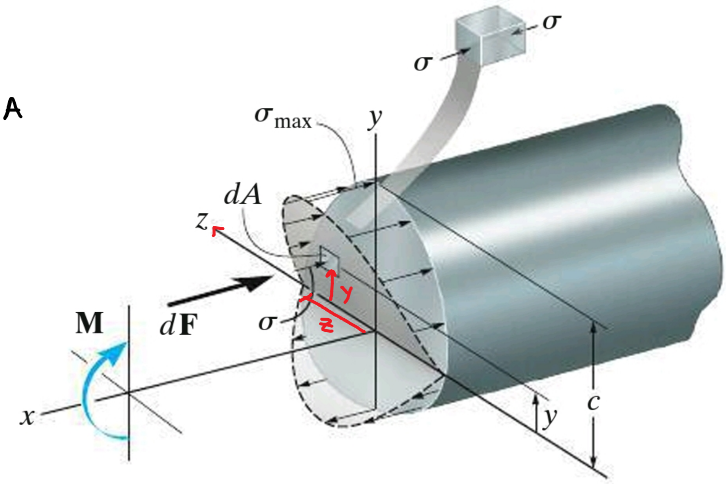


Bending Stress variation

The first moment of the member's cross-sectional area about the neutral axis must be zero. This condition can only be satisfied if the neutral axis is also the horizontal centroidal axis of the C/s. Recall that the location of centroid of a C/s area is defined as $\bar{y} = \frac{\int y \, dA}{\int dA}$. So $\int y \, dA = 0$ means $\bar{y} = 0$

Radius of curvature of neutral line

We also have to find R. We first obtain an expression for the moment (about the center of the C/s) due to internal traction in the cross-sectional plane



Moment in any c/s about z

$$M_z = \int_{A_0} \left\{ (y\hat{j} + z\hat{k}) \times (\sigma_{xx}\hat{i}) \right\} \cdot \hat{k} \, dA$$

$$= -\int_{A_0} y \, \sigma_{xx} \, dA$$

$$= \int_{A_0} \frac{E}{R} y^2 \, dA$$

$$= \frac{E}{R} \int_{A_0} y^2 \, dA$$

This integral is called the moment of inertia of the C/s area about neutral axis (or the z-axis) denoted by I_{zz} } is a property related to the geometry of the cross-section

Flexural rigidity

Thus, we finally get: $M_z = \dfrac{E I_{zz}}{R} \Rightarrow R = \dfrac{E I_{zz}}{M_z}$

The product of the Young's modulus and the moment of inertia I_{zz} is called the flexural rigidity of a beam. The radius of curvature is larger for a higher value of EI_{zz}.

Smaller EI_{zz} → smaller R → more curvature

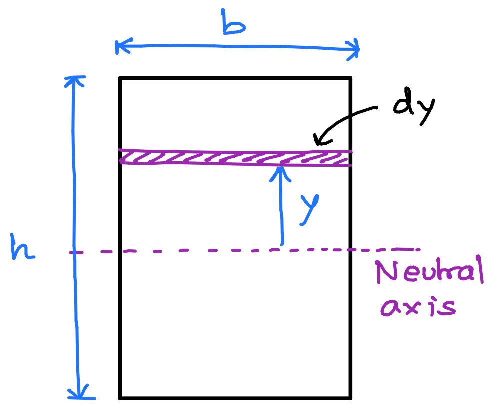
Higher EI_{zz}

larger R → smaller curvature

Final expression of bending stress σ_{xx}

$$\sigma_{xx} = -E\frac{y}{R} = -Ey\frac{M_z}{EI_{zz}} = -\frac{M_z y}{I_{zz}}$$

I_{zz} for a rectangular cross-section



Consider a small strip of thickness dy at a height y above the neutral axis

The moment of inertia of area about the neutral axis is calculated as :

$$I_{zz} = \iint y^2\, dA = \int y^2\, b\,dy = b\left.\frac{y^3}{3}\right|_{-h/2}^{h/2} = \frac{bh^3}{12}$$

So, moment of inertia of C/s area increases with increase in breadth and height of the beam C/s, but is more sensitive to 'h'. We can imagine two rectangular beams made of same material and same C/s area with the first one having smaller height h than the second one. Because of larger h, the 2nd beam will have larger I_{zz} and thus a larger flexural rigidity when compared to the first beam. So it will be more difficult to bend the second beam although both the beams have same C/s area.