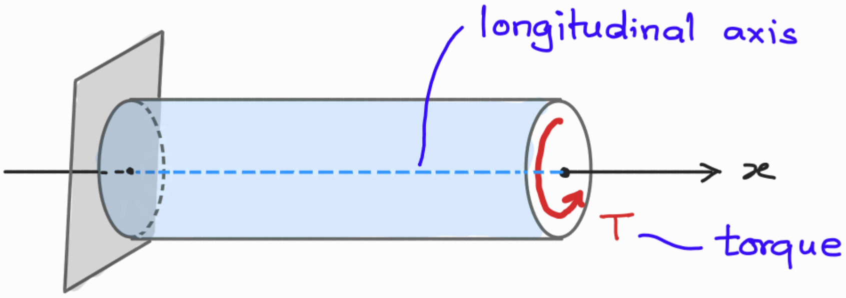
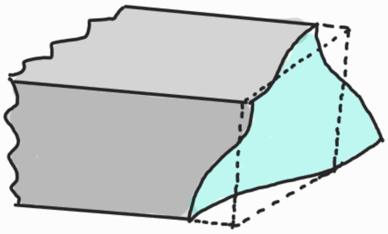
Torsion of circular members

Torque is a moment that tends to twist a member abt its longitudinal axis.

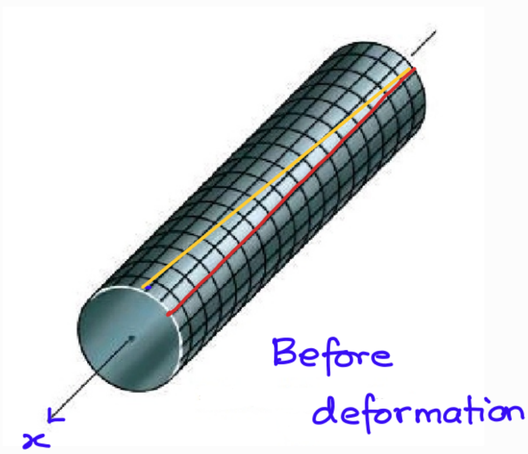


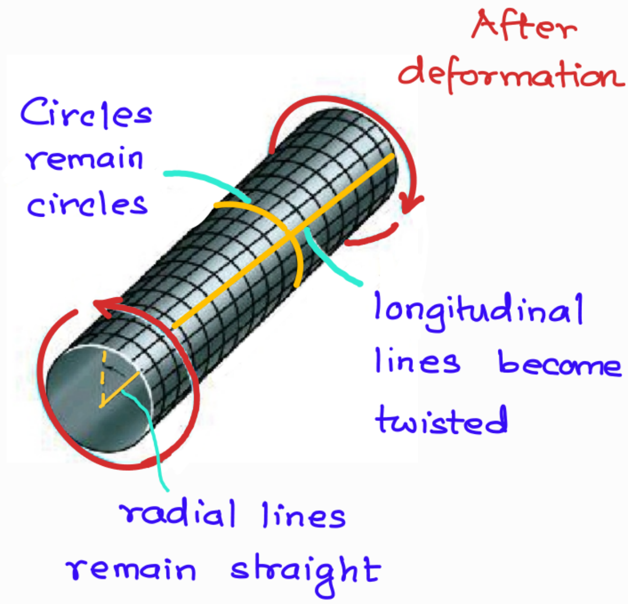
The theory of torsion we talk about here concerns torques which twist the members but do not induce any warping, that is, plane cross-sections which are perpendicular to the axis of the member remain plane after twisting



Warped C/s (occurs in most non-circular c/s under torque)

Furthermore, radial lines remain straight and rotate as the C/s rotates

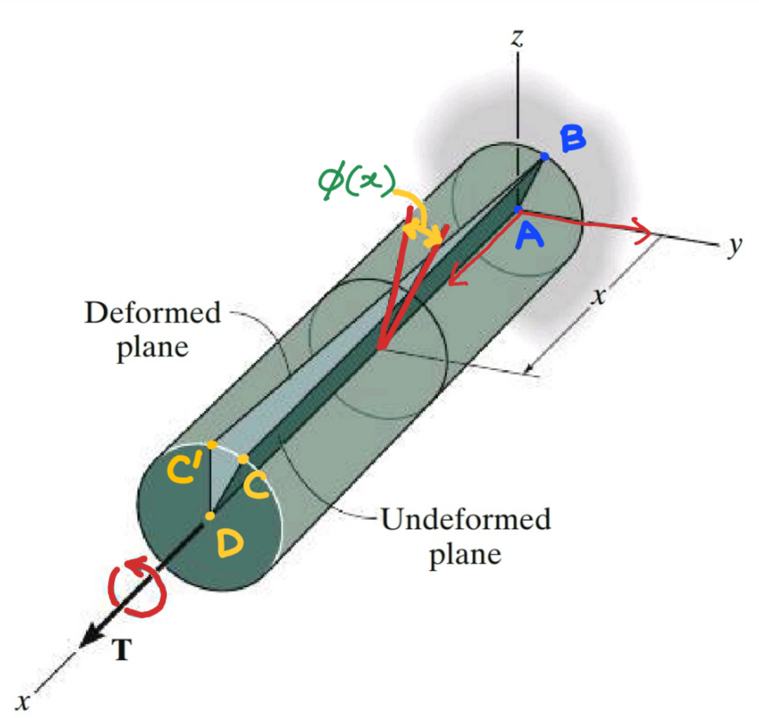




For example, consider the following member fixed at one end and subject to a torque T at the other end.

The torque shown is positive (right-hand-thumb rule).

The member twists under the action of the torque and the radial plane ABCD moves to ABC′D



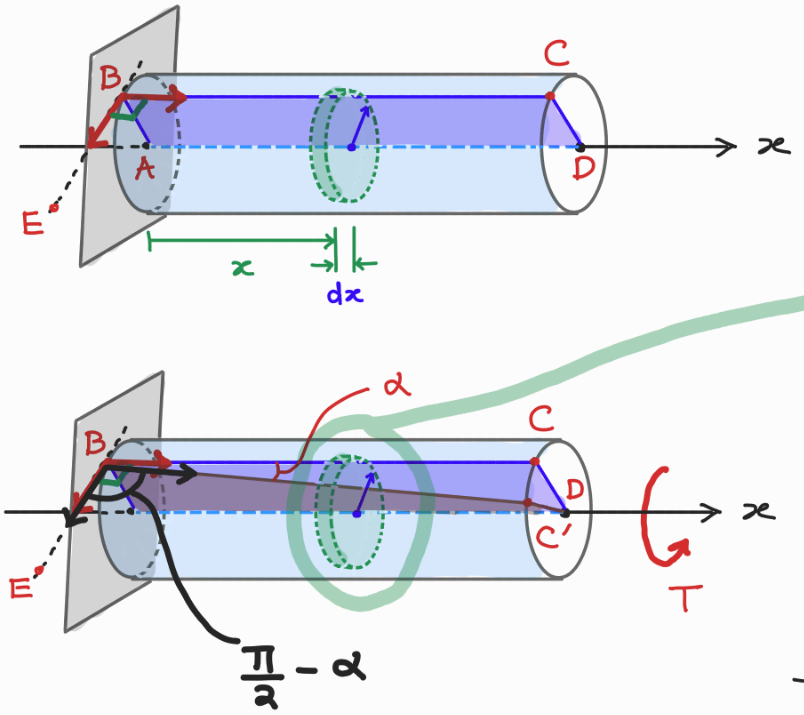
While in the case of axial loading the measure of deformation was axial elongation, here an appropriate measure is the amount by which the member twists at a given cross-section, the rotation angle $\phi(x)$

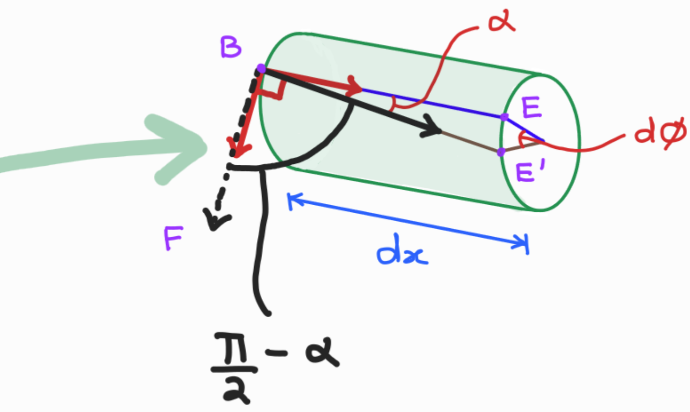
This rotation angle varies along the length of the member and is called the **angle of twist**, ϕ.

Sign convention : ϕ is +ve in the same direction as +ve torque, and vice versa

Further, much like normal strain ε_{xx} was the measure of strain in axial loading of a member, here it will be the engineering shear strain γ_{xy} (more appropriately $\gamma_{z\theta}$ when using r-θ-z csys). A relationship between γ (dropping the subscript) and ϕ will be established next.

Let us consider an infinitesimal cylinder of the shaft at a distance x from the left end with an arbitrary radius r





The relative rotation between two adjacent cross-sections (at distance dx) is $d\phi$ (infinitesimal angle of twist)

Note that the shear strain is defined by a change in the original right angle between two small line elements In this case, two line elements: one in the direction of the axis BE and other in the direction of tangent to the circular c/s BF. After deformation, BE gets oriented along BE'

As such, the change in angle is α

Engineering shear strain, γ, induced by this deformation can be obtained from the length of arc EE':

$$\text{arc } EE' = \alpha \, dx = r \, d\phi$$

$$\Rightarrow \quad \alpha = r \frac{d\phi}{dx}$$

$$\text{Shear strain} \equiv \gamma = \alpha = r \frac{d\phi}{dx} \qquad \text{(Strain-displacement)}$$

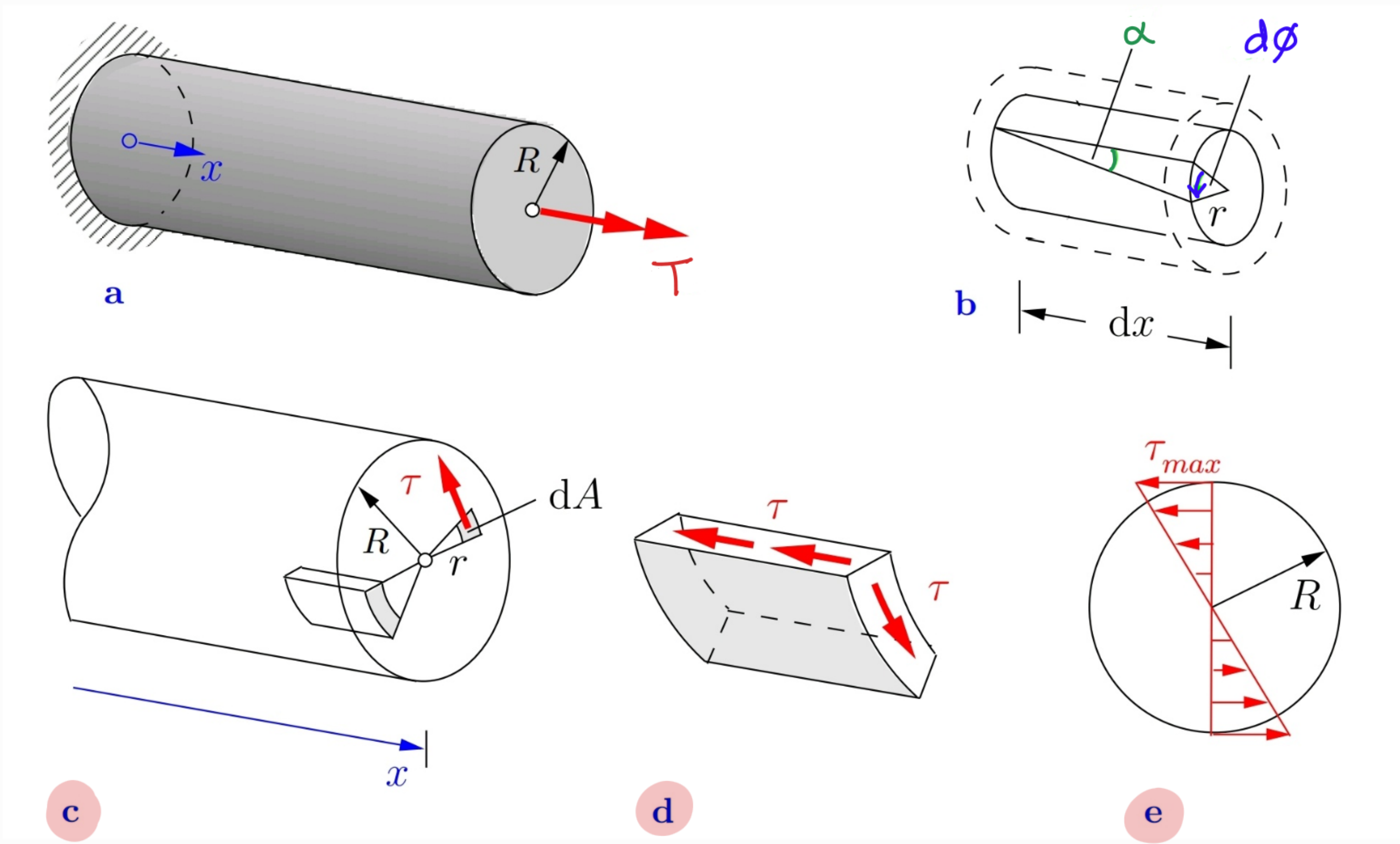
Note that the shear strain varies linearly along the radius of the cross-section.

For linear elastic isotropic material, the shear stress τ is obtained from Hooke's law:

$$\tau = G\gamma = G r \frac{d\phi}{dx} \qquad \text{(Stress-strain)}$$

Hence, the shear stress also varies linearly from zero at $r=0$ to a maximum value at the outer surface $r=R$ of a circular shaft.

The shear stresses are tangential to the cross-sectional surface and also perpendicular to any radial line on the cross-section. They are depicted in the following figure (c), (d), and (e)



Equilibrium relation

One should notice that there is no resultant force on any cross-section, only a net torque. Thus, the integration of the shear stress over any given C/s should yield zero net force, which is evident from the stress distribution cancel-ling each other in figure (e)

The net torque T over a cross-section is obtained by taking moments of the shear force about the axis $(r=0)$

$$T = \int r\,(\tau\, dA) = \int r\; G\, r\, \frac{d\phi}{dx}\, dA$$

$$\Rightarrow \quad T = G\,\frac{d\phi}{dx} \boxed{\int r^2\, dA}$$

$$J := \int r^2\, dA$$ = Second moment of area

defined as

(also called Polar moment of inertia)

↓

is a purely geometry of C/s related property (not to be confused with mass moment of inertia)

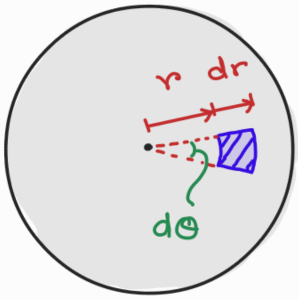
$$\Rightarrow \boxed{T = GJ\,\frac{d\phi}{dx}}$$

Torsional rigidity

For a solid circular shaft, the polar moment of inertia is:

$$J = \int r^2\, dA$$

$dA = r\,d\theta\,dr$



$$= \int_0^{2\pi}\int_0^{R} r^2\, r\, dr\, d\theta$$

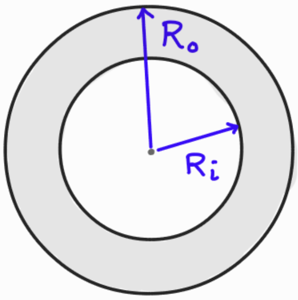
$$= 2\pi\left(\frac{R^4}{4}\right) = \frac{\pi R^4}{2} \quad \left(= \frac{\pi D^4}{32}\right)$$

diameter

For a hollow circular shaft, the polar moment of inertia is:

$$J = \int r^2\, dA$$

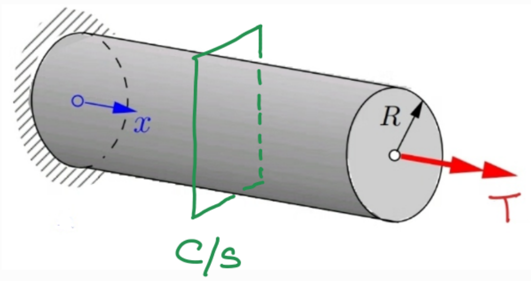
$dA = r\,d\theta\,dr$



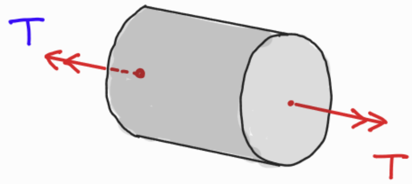
$$= \int_0^{2\pi}\int_{R_i}^{R_o} r^2\, r\, dr\, d\theta$$

$$= 2\pi\,\frac{R_o^4 - R_i^4}{4} = \frac{\pi}{2}\left(R_o^2 - R_i^2\right)$$

Total angle of twist



If we consider a circular shaft clamped at one end and subjected to a torque T at the free end, then, in each arbitrary C/s ⊥ to the x-axis, the net internal torque is equal to T, which is constant over the length L of the member



The total angle of twist $\phi(L)$ at the free end is:

$$\int_0^L \frac{d\phi(x)}{dx}\,dx = \int_0^L \frac{T}{GJ}\,dx$$

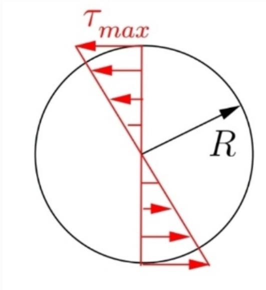
$$\Rightarrow \quad \phi(L) - \underbrace{\cancel{\phi(0)}^{\,0}}_{\text{clamped}} = \frac{TL}{GJ} \quad \Rightarrow \quad \boxed{\phi(L) = \frac{TL}{GJ}}$$

If we eliminate $\frac{d\phi}{dx}$ from the formula $T = GJ\frac{d\phi}{dx}$ using the stress-strain relation $\tau = G r \frac{d\phi}{dx}$, we obtain the torsion formula, which gives the distribution of the shear stress:

$$\boxed{\tau = \frac{T}{J}\,r}$$

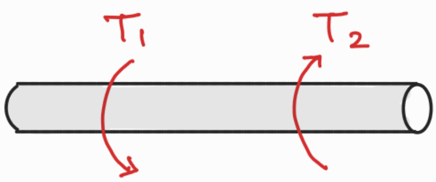
The maximum value appears at the outer boundary $r=R$

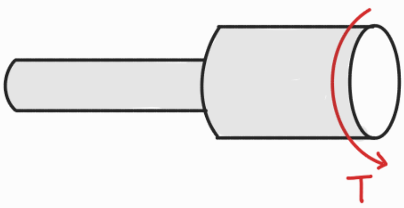
$$\tau_{max} = \frac{T}{J} R$$



Multiple torques

If a shaft is subjected to multiple torques or the c/s area or the shear modulus changes abruptly from one region of the shaft to the next, such as shown below,





then the formula $\phi(L) = \frac{TL}{GJ}$ should be applied to each segment of the shaft where these quantities are all constant.

The angle of twist of one end of shaft w.r.t the other end is found by algebraic addition of the angles of twist of each segment

$$\boxed{\phi = \sum_i \frac{T_i L_i}{G_i J_i}}$$