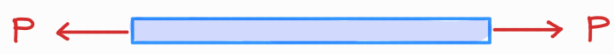
Applications of Linear Elasticity

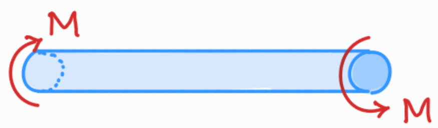
The linear elastic model was introduced in the previous lectures

We will now look at mechanics problems and their solutions using isotropic linear elastic constitutive material model:

* Axial loading & deformation
  - geometry is of a long slender bar and load acts along the length of the bar



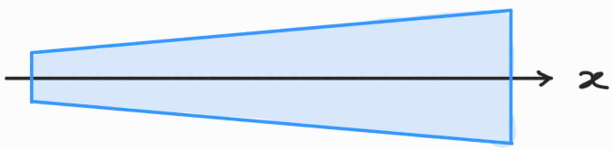
* Torsion of long slender circular bar
  - load acts to twist the bar



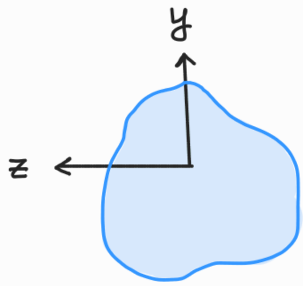
* Bending of straight beams
  - geometry is of a long and slender beam
  - load is transverse to the beam length

Axial loading and deformation

We consider long and thin straight members loaded in a way that it deforms predominantly in the axial direction. The x-axis is taken as the longitudinal axis, with its c/s lying in the y-z plane



longitudinal axis of a slender member



cross-section

Any static analysis of a structural component involves the following three considerations:

(1) Equilibrium

(2) Constitutive relation

(3) Kinematics (or Compatibility)

Here, (2) is taken to be isotropic linear elastic solid.

As for (1) and (3), it is assumed that the only significant stresses and strains occur in the axial direction, and so the 6 stress-strain relations reduce to just <u>one</u> relation

$$\sigma_{xx} = E\,\epsilon_{xx}$$

or, dropping the subscripts,

$$\sigma = E \epsilon$$

In axial deformation, it is assumed that before and after deformation:

a) the axis of the member remains straight, and

b) the c/s that were initially perpendicular remains perpendicular after deformation.

This implies that, although the strain might vary along the axis, it remains constant over any cross-section

Generally, undeformed configurations are used for writing equilibrium equations since the deformations are small.

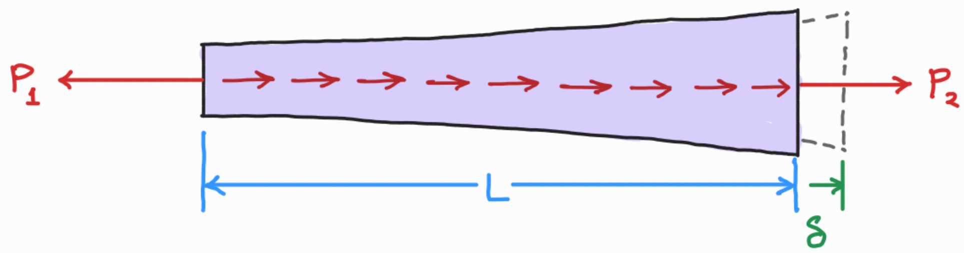
Statically Determinate vs. Statically Indeterminate Members

In statically determinate cases, the three sets of equations (i.e., equilibrium, constitutive, and kinematics) can be solved independently of each other in a decoupled fashion. This is, however, not the case with statically indeterminate members.

Thermo-elastic deformation of an axially loaded bar

We now develop an equation that can be used to determine the linear elastic displacement of a bar subjected to axial loads under increase of temperature.

Let's consider this bar with c/s area that varies with length and is made of a material that has variable Young's modulus along the length.

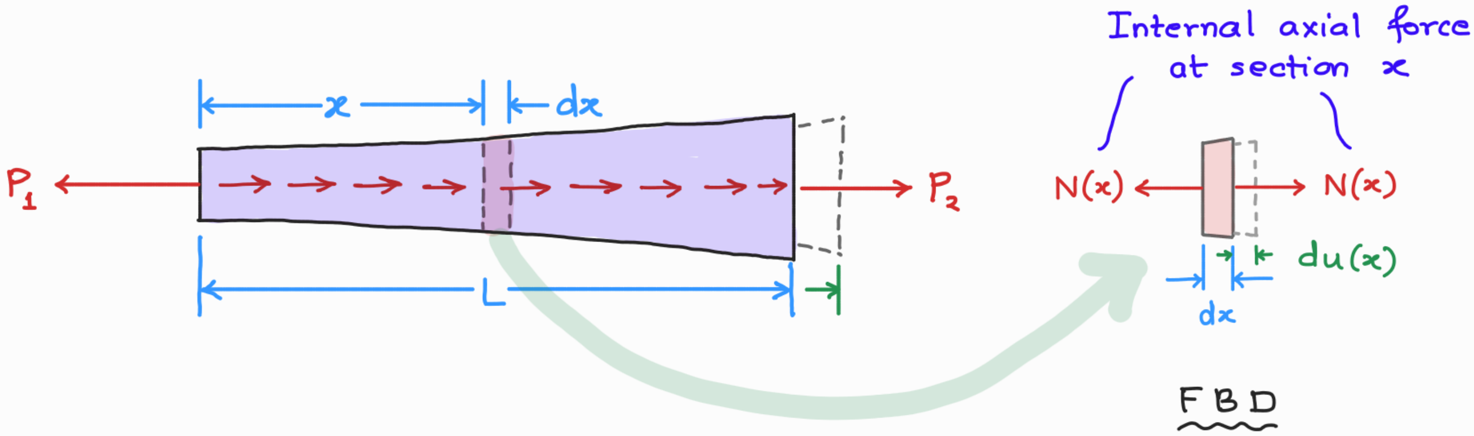


Bar is subjected to concentrated load at its ends and an ext. variable distributed load along its length (that could be its weight in a vertical position) and variable temperature increase.

Find the relative displacement u (delta) of one end of the bar w.r.t. the other end as caused by the loading.

Isolate a differential element of length dx and c/s area $A(x)$ from the bar at an arbitrary position x, where the Young's modulus is $E(x)$. Draw the free-body-diagram (FBD)

Let the relative displacement of this element be $du(x)$



a) Equilibrium rel: $\sigma(x) = \dfrac{N(x)}{A(x)}$

coeff of thermal expansion

b) Constitutive rel: $\epsilon(x) = \dfrac{\sigma(x)}{E(x)} + \alpha(x)\,\Delta T(x)$

c) Kinematics rel: $\dfrac{du(x)}{dx} = \epsilon(x) \Rightarrow u = \displaystyle\int_0^L \epsilon(x)\,dx$

Putting all of them together, we get:

$$u = \underbrace{\int_0^L \frac{N(x)}{A(x)\,E(x)}\,dx}_{\text{Elastic deformation}} + \underbrace{\int_0^L \alpha(x)\,\Delta T(x)\,dx}_{\text{Thermal deformation}}$$

* Special case of constant load and C/s area

In many cases, the bar will have

a) a constant C/s area A

b) the material will be homogeneous $\Rightarrow$ E is constant.

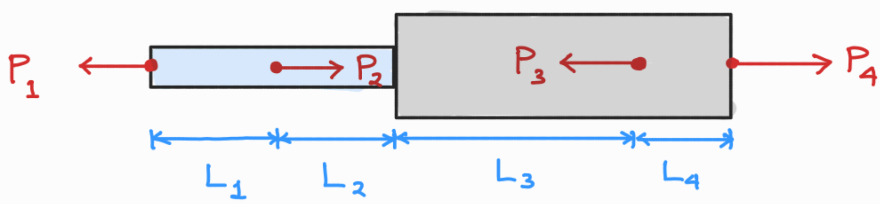
c) a constant external force applied at each end and no distributed force $\Rightarrow$ $N(x) = P$ (constant)

d) no temperature change

The relative displacement for such a case would be

$$\boxed{u = \frac{PL}{AE}}$$

Further, if the bar is subjected to several different axial forces along its length, or the C/s area or Young's modulus changes abruptly from one region of the bar to the next,



then the above formula can be applied to each segment of the bar where these quantities remain constant. The total relative displacement of the bar can be found by addition of the relative displacements of the ends of each segment.

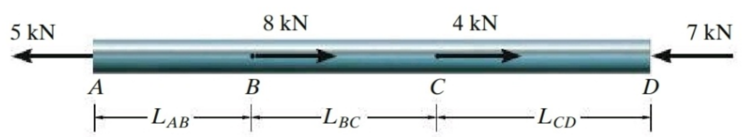
$$u = \sum_i \frac{P_i L_i}{A_i E_i}$$

Sign Convention

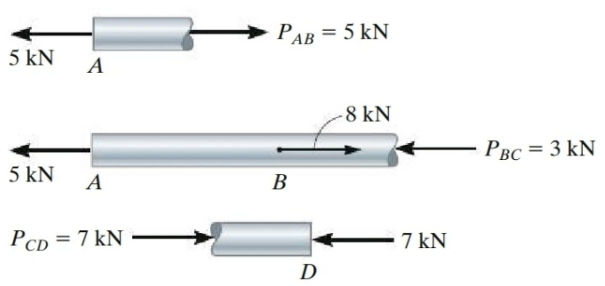
Tensile force is +ve & $u \rightarrow$ +ve if elongated

Compressive force is −ve & $u \rightarrow$ − ve if contracted

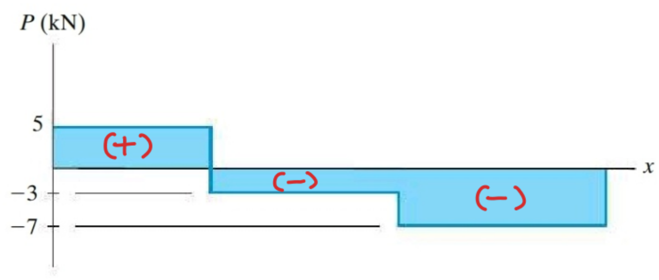
Ex:



(a)



(b)



(c) Axial force diagram