Introduction to Constitutive Relations

Recall that we are need of 6 more equations to solve for 15 unknowns for a general three dimensional mechanics problem. Unknowns → 6 stresses, 6 strains, 3 displacements

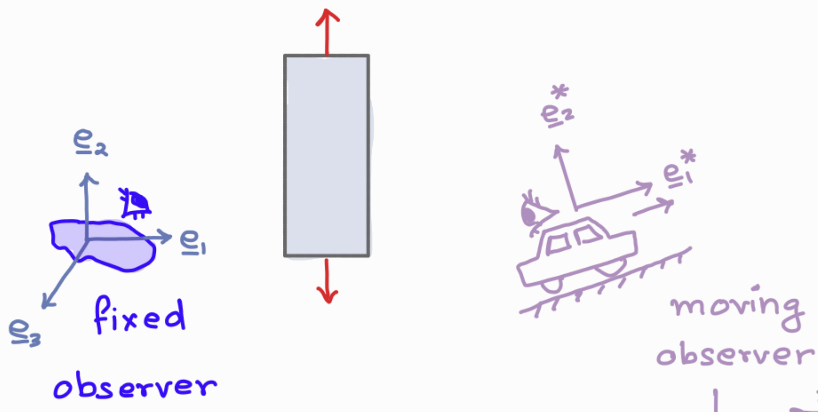
Equations → 3 stress-equilibrium, 6 strain-disp rels

Constitutive relations, or typically called stress-strain relations in solid mechanics, give us 6 more relations

As stress and strain tensors are defined at a point, the constitutive relations are local, in the sense that pointwise constitutive relations are established from global experimental tests on a full specimen. Hence we need to conduct experiments on homogeneous specimens with uniform geometry, loading and BCs to ensure that the stress $\underline{\underline{\sigma}}(\underline{x})$ and strain $\underline{\underline{\epsilon}}(\underline{x})$ fields are almost uniform

* Principle of material objectivity (or ref. frame indifference)

A material's stress-strain behavior is independent of the observer's frame of reference. Violating this property would imply that material behavior could be altered by simply moving the reference frame (or csys) or by rotating the material, which is physically not true.



By material objectivity,

$$f_1(\cdot) = f_2(\cdot)$$

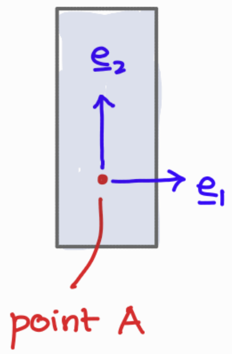
$\hookrightarrow \sigma = f_1(\epsilon)$

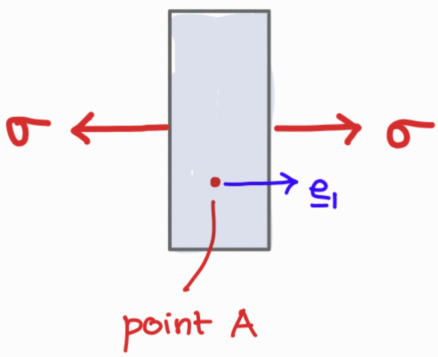
$\hookrightarrow \sigma^* = f_2(\epsilon^*)$

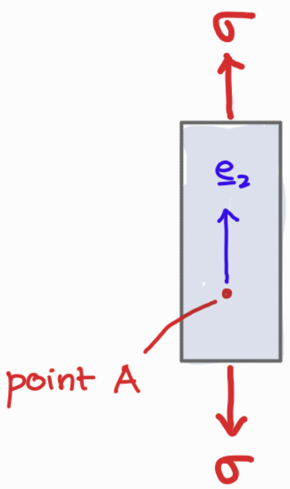
the measured variables can change for two observers

Isotropic : A material is called isotropic at a point if the material has same properties in ALL directions

In other words, the material constants $\underline{\theta}$ will not depend on the orientation of csys at that point







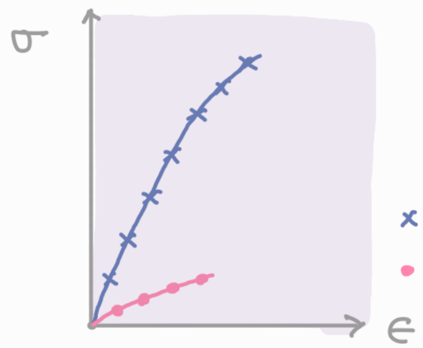
(True Stress-strain relation at pt A)

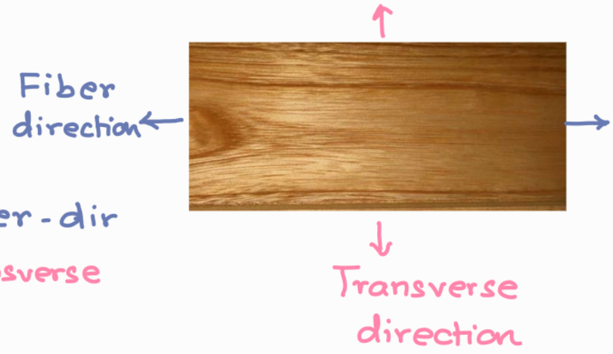
$$\sigma_A = f_{\underline{\theta}}(\epsilon_A) \qquad \sigma_A^* = f_{\underline{\theta}^*}(\epsilon_A^*)$$

Isotropy $\Rightarrow$ $\boxed{\underline{\theta} = \underline{\theta}^*}$

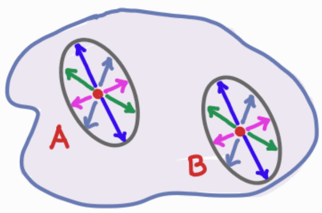
Anisotropic: A material whose material constants are direction dependent e.g. wood

Wood is much stronger along its long-itudinal fiber orientation

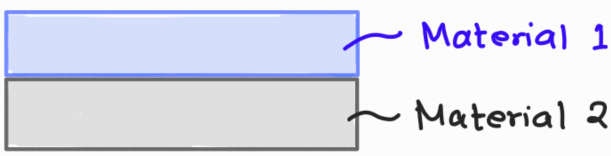




Homogeneous: A material is said to be homogeneous if the material properties are spatially invariant. In other words, the stress-strain behavior is same at all material points, but can still have directional dependence at a point.



Most engineering materials at the continuum macroscopic level are taken to be homogeneous.



Isotropic but inhomogeneous bimetallic plate

Not isotropic but can be considered homogeneous

Linear Elastic Constitutive model

The simplest constitutive relation for solid materials is the linear elastic model, which turns out to be a very good prediction model of the response of components which undergo small deformations e.g. steel and concrete structures under daily operational loads.

Linear elastic model can be used to describe materials which respond as follows:

1) Strains are small (small strain assumption is valid)

2) Stress is proportional to strain $\sigma \propto \epsilon$ (linear)

3) Material returns to its original shape when load is removed, and the unloading path is same as the loading path (elastic, but not viscoelastic)

4) there is no strain rate or loading rate dependence (elastic, but not viscoelastic)

This model represents engineering materials upto their proportionality limit.

Generalized Hooke's Law

For linear elastic solid σ_{ij} is a linear function of ϵ_{ij}

$$\Rightarrow \quad \sigma_{ij} = C_{ijkl}\,\epsilon_{kl}$$

$C_{ijkl} \rightarrow i = 1, 2, 3; \quad j = 1, 2, 3; \quad k = 1, 2, 3; \quad l = 1, 2, 3$

↳ total constants $3 \times 3 \times 3 \times 3 = 3^4 = 81$ constants

However, there are only six independent stress and strain components. Therefore, we should only fit the relation betn the independent components.

$$\left.\begin{aligned} \sigma_{ij} &= \sigma_{ji} \\ \epsilon_{kl} &= \epsilon_{lk} \end{aligned}\right\} \Rightarrow \quad C_{ijkl} = C_{jilk}$$

$$81 \xrightarrow[\text{symmetry}]{\text{Minor}} 36$$

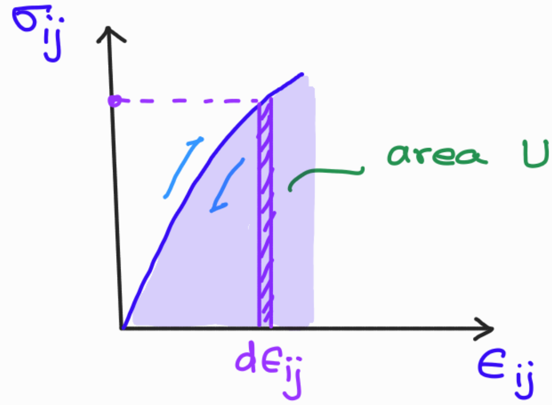
In matrix form, this looks as follows:

$$\underbrace{\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{bmatrix}}_{\text{Voigt notation}} = \underbrace{\begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1113} & C_{1112} \\ C_{2211} & C_{2222} & C_{2233} & C_{2223} & C_{2213} & C_{2212} \\ C_{3311} & C_{3322} & C_{3333} & C_{3323} & C_{3313} & C_{3312} \\ C_{2311} & C_{2322} & C_{2333} & C_{2323} & C_{2313} & C_{2312} \\ C_{1311} & C_{1322} & C_{1333} & C_{1323} & C_{1313} & C_{1312} \\ C_{1211} & C_{1222} & C_{1233} & C_{1223} & C_{1213} & C_{1212} \end{bmatrix}}_{\text{36 unknown constants}} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \epsilon_{23} \\ \epsilon_{13} \\ \epsilon_{12} \end{bmatrix}$$

Voigt notation of writing down a symmetric matrix as a vector

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ & a_{22} & a_{23} \\ & & a_{33} \end{bmatrix} \longrightarrow \begin{bmatrix} a_{11} \\ a_{22} \\ a_{33} \\ a_{23} \\ a_{13} \\ a_{12} \end{bmatrix}$$

Elastic energy is the area under the elastic portion of stress-strain curve



scalar

$$dU = \sigma_{ij} \, d\epsilon_{ij}$$

$$\Rightarrow \quad \sigma_{ij} = \frac{dU}{d\epsilon_{ij}}$$

Also, $\sigma_{ij} = C_{ijkl} \, \epsilon_{kl}$

$$\therefore \quad \frac{\partial \sigma_{ij}}{\partial \epsilon_{kl}} = \frac{\partial^2 U}{\partial \epsilon_{kl} \, \partial \epsilon_{ij}} = C_{ijkl}$$

Also, we can interchange the indices 'ij' with 'kl' and get:

$$\frac{\partial^2 U}{\partial \epsilon_{ij} \, \partial \epsilon_{kl}} = \frac{\partial^2 U}{\partial \epsilon_{kl} \, \partial \epsilon_{ij}} \Rightarrow C_{ijkl} = C_{klij}$$

Major symmetry

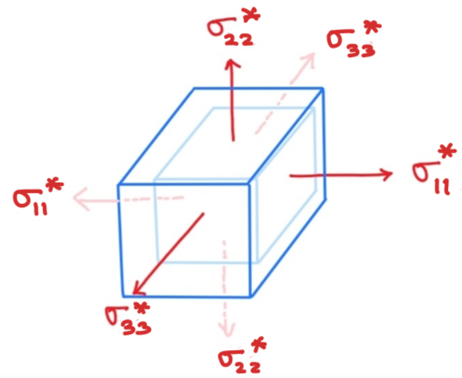
After major symmetry, the linear elastic constitutive relation looks as follows:

$$\underset{\text{Voigt notation}}{\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \tau_{23}\\ \tau_{13}\\ \tau_{12}\end{bmatrix}} = \underbrace{\begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1113} & C_{1112} \\ & C_{2222} & C_{2233} & C_{2223} & C_{2213} & C_{2212} \\ & & C_{3333} & C_{3323} & C_{3313} & C_{3312} \\ & \text{SYM} & & C_{2323} & C_{2313} & C_{2312} \\ & & & & C_{1313} & C_{1312} \\ & & & & & C_{1212} \end{bmatrix}}_{\text{21 unknown constants}} \begin{bmatrix}\epsilon_{11}\\ \epsilon_{22}\\ \epsilon_{33}\\ \epsilon_{23}\\ \epsilon_{13}\\ \epsilon_{12}\end{bmatrix}$$

Thus, for an ANISOTROPIC LINEAR ELASTIC solid material, the stress-strain relation has 21 independent elastic constants.

Isotropic linear elastic materials have just 2 independent elastic constants.

If we consider a small cuboidal material made of isotropic linear elastic material and is subjected to principal stresses on its faces, the cuboid will remain cuboidal after deformation (no distortion / shape change)



Thus, the normals to these faces must coincide with the principal strain directions.

Hence, for an isotropic material, one can relate the principal stresses σ_{11}^*, σ_{22}^*, σ_{33}^* with the three principal strains ϵ_{11}^*, ϵ_{22}^*, ϵ_{33}^* through suitable elastic constants

Along the principal direction (say $\underline{e}_1^*$), we then have:

$$\sigma_{11}^* = C_{1111}\,\epsilon_{11}^* + C_{1122}\,\epsilon_{22}^* + C_{1133}\,\epsilon_{33}^*$$

We note that C_{1122} & C_{1133} must be equal since the effect of σ_{11}^* in the directions of $\underline{e}_2^*$ and $\underline{e}_3^*$ (which are perpendicular to $\underline{e}_1^*$) must be the same for an isotropic material. Hence, for σ_{11}^*, we can write

$$\sigma_{11}^* = C_{1111}\,\epsilon_{11}^* + C_{1122}\,(\epsilon_{22}^* + \epsilon_{33}^*)$$

$$= \underbrace{(C_{1111} - C_{1122})}_{2\mu}\,\epsilon_{11}^* + \underbrace{C_{1122}}_{\lambda}\,\underbrace{(\epsilon_{11}^* + \epsilon_{22}^* + \epsilon_{33}^*)}_{\text{1st invariant of strain } J_1}$$

$$\Rightarrow \quad \sigma_{11}^* = \lambda\, \mathrm{tr}(\underline{\underline{\epsilon}}) + 2\mu\,\epsilon_{11}^*$$

LAME'S CONSTANTS

For a general coordinate system, it can be shown that the six stress components $\sigma_{11}, \sigma_{22}, \sigma_{33}, \tau_{12}, \tau_{13}, \tau_{23}$ are related to six strain components $\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, \epsilon_{12}, \epsilon_{13}, \epsilon_{23}$

$$\sigma_{11} = \lambda \, tr(\underline{\underline{\epsilon}}) + 2\mu \, \epsilon_{11}$$

$$\sigma_{22} = \lambda \, tr(\underline{\underline{\epsilon}}) + 2\mu \, \epsilon_{22}$$

$\lambda, \mu \rightarrow$ Lame's constants

$$\sigma_{33} = \lambda \, tr(\underline{\underline{\epsilon}}) + 2\mu \, \epsilon_{33}$$

$$\tau_{12} = 2\mu \, \epsilon_{12}$$

$$\tau_{13} = 2\mu \, \epsilon_{13}$$

$$\tau_{23} = 2\mu \, \epsilon_{23}$$

2 elastic constants for Isotropic Linear Elastic material

Alternatively, we can use another form where strain is expressed in terms of stress. It is also called three-dimensional Hooke's law of isotropic linear elasticity:

$$\boxed{\underline{\underline{\epsilon}} = \frac{1+\nu}{E} \underline{\underline{\sigma}} - \frac{\nu}{E} tr(\underline{\underline{\sigma}}) \, \underline{\underline{I}}}$$

$$\epsilon_{11} = \frac{1}{E}\left(\sigma_{11} - \nu(\sigma_{22} + \sigma_{33})\right)$$

$$\epsilon_{22} = \frac{1}{E}\left(\sigma_{22} - \nu(\sigma_{11} + \sigma_{33})\right)$$

$$\epsilon_{33} = \frac{1}{E}\left(\sigma_{33} - \nu(\sigma_{11} + \sigma_{22})\right)$$

In this representation, we have three constants:

E — Young's Modulus

ν — Poisson's Ratio

G — Shear Modulus

$$\gamma_{12} = 2\epsilon_{12} = \frac{\tau_{12}}{G}$$

$$\gamma_{13} = 2\epsilon_{13} = \frac{\tau_{13}}{G}$$

$$\gamma_{23} = 2\epsilon_{23} = \frac{\tau_{23}}{G}$$

However, there are only two indep. constants. Therefore, we have

$$\boxed{G = \frac{E}{2(1+\nu)}}$$

Physical Significance of E, G and ν

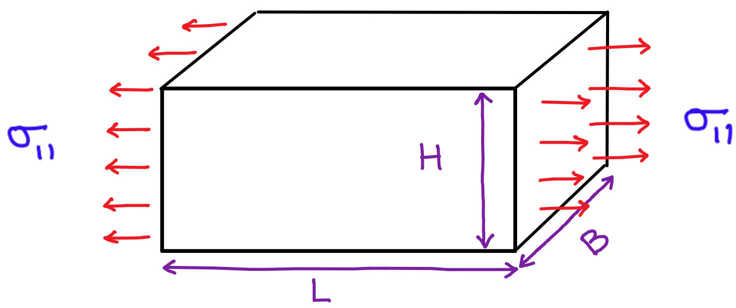
$$\epsilon_{11} = \frac{1}{E}\left(\sigma_{11} - \nu\left(\sigma_{22} + \sigma_{33}\right)\right)$$

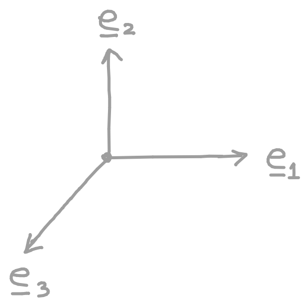
$$\epsilon_{22} = \frac{1}{E}\left(\sigma_{22} - \nu\left(\sigma_{11} + \sigma_{33}\right)\right)$$

$$\epsilon_{33} = \frac{1}{E}\left(\sigma_{33} - \nu\left(\sigma_{11} + \sigma_{22}\right)\right)$$

We can see that the normal strain in one direction not only depends on the normal stress in that direction but also on the normal stresses in other two directions.

To understand the physical relevance, we can think of a tensile testing. Suppose that we have a rectangular beam of length L, breadth B and height H.



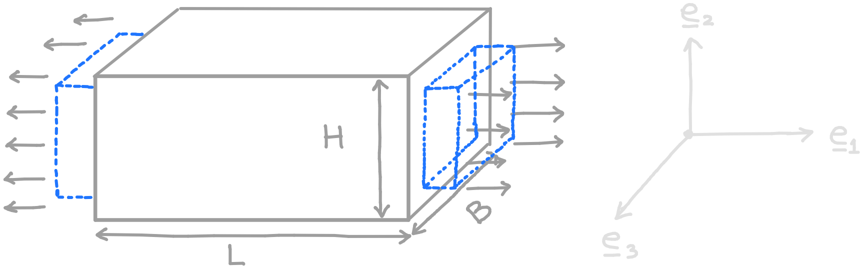


The beam is oriented such that its length is along $\underline{e}_1$, its height is along $\underline{e}_2$ and its breadth is along $\underline{e}_3$.

Let's apply force on the left and right faces to stretch the beam. So σ_{11} will be non-zero while the shear components τ_{12} and τ_{13} will be zero. Also, we are not applying any force on the lateral surface ($\underline{e}_2$ and $\underline{e}_3$ planes), so there is no stress on them. Infact, any internal section with normal along $\underline{e}_2$ & $\underline{e}_3$ will not have any traction component.

The state of stress in this case will then be

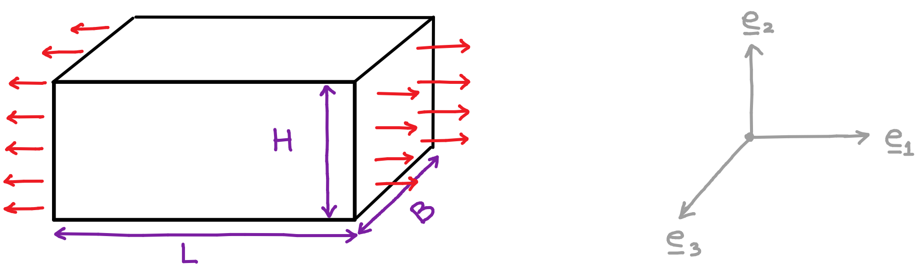
$$\begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



This stress will lead to some strain in the body. We know that ϵ_{11} is generated because the length of the beam changes. However, ϵ_{22} and ϵ_{33} are also generated: due to stretching in one direction, there will be contraction in other two directions. But there will be no shear strain generated if we are careful in stretching the body uniformly. Thus, the state of strain for the rectangular beam will be

$$\begin{bmatrix} \epsilon_{11} & 0 & 0 \\ 0 & \epsilon_{22} & 0 \\ 0 & 0 & \epsilon_{33} \end{bmatrix}$$

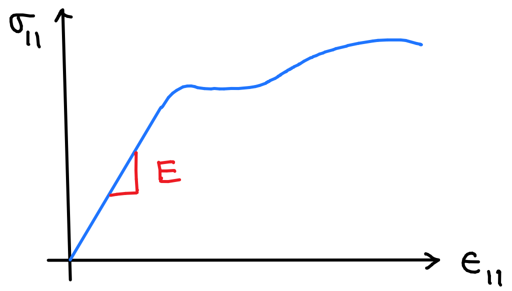
(1) Young's Modulus (E)

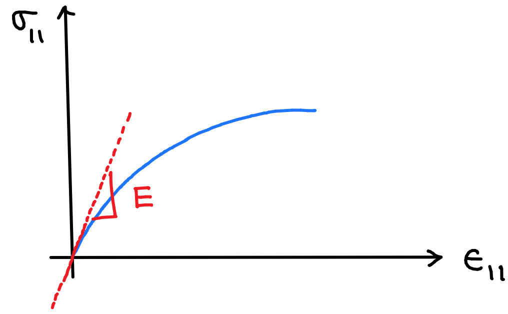


If the elongation is uniform along the length of the bar, the local strain will be equal to the average normal strain. Thus,

$$\epsilon_{11} = \frac{\Delta L}{L}, \qquad \epsilon_{22} = \frac{\Delta H}{H}, \qquad \epsilon_{33} = \frac{\Delta B}{B}$$

If we now draw the stress-strain curve of $\sigma_{11} - \epsilon_{11}$ from a tensile test experiment by measuring the change in length, we may get a curve



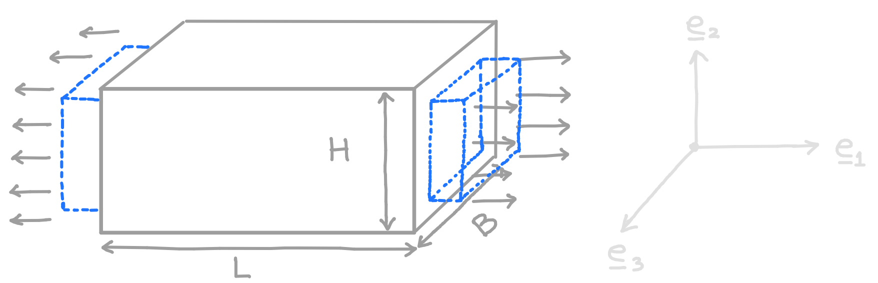


The initial slope of this graph gives us the Young's modulus

$$E = \left. \frac{\partial \sigma_{11}}{\partial \epsilon_{11}} \right|_{\epsilon_{11}=0}$$

While computing this derivative from the graph, we should not have σ_{22} or σ_{33} present. Essentially, the rectangular beam should be stretched in such a way that it can freely shrink in the lateral directions.

For a uniaxial tensile test experiment, we can say that



$$\sigma_{22} = \sigma_{33} = 0$$

$$\sigma_{11} = E\,\epsilon_{11}$$

(for small ϵ_{11})

(2) Poisson's ratio (ν)

The Poisson's ratio of a material is defined as

$$\nu = - \frac{\text{Lateral normal strain}}{\text{Longitudinal normal strain}}$$

In a uniaxial tensile test, we directly impose longitudinal normal strain in the e_1-direction and this, in turn, induces strain along e_2 and e_3 directions. For an isotropic body, the lateral strains in these directions will be equal. Thus, the Poisson's ratio from a uniaxial tensile experiment would be

$$\nu = -\frac{\epsilon_{22}}{\epsilon_{11}} = -\frac{\epsilon_{33}}{\epsilon_{11}}$$

We can also derive the Poisson's ratio from the stress-strain relation

In the tensile test,

$$\sigma_{22} = \sigma_{33} = 0$$

- $\epsilon_{11} = \frac{1}{E}\left(\sigma_{11} - \nu\left(\cancelto{0}{\sigma_{22}} + \cancelto{0}{\sigma_{33}}\right)\right)$

  $\Rightarrow \quad \epsilon_{11} = \sigma_{11}/E$

- $\epsilon_{22} = \frac{1}{E}\left(\cancelto{0}{\sigma_{22}} - \nu\left(\sigma_{11} + \cancelto{0}{\sigma_{33}}\right)\right)$

  $\Rightarrow \quad \epsilon_{22} = -\frac{\nu}{E}\sigma_{11} = -\nu\,\epsilon_{11}$

- $\epsilon_{33} = \frac{1}{E}\left(\cancelto{0}{\sigma_{33}} - \nu\left(\sigma_{11} + \cancelto{0}{\sigma_{22}}\right)\right)$

  $\Rightarrow \quad \epsilon_{33} = -\frac{\nu}{E}\sigma_{11} = -\nu\,\epsilon_{11}$

We get the same formula

(3) Shear Modulus (G)

$$\gamma_{12} = 2\epsilon_{12} = \frac{\tau_{12}}{G}$$

$$\gamma_{13} = 2\epsilon_{13} = \frac{\tau_{13}}{G}$$

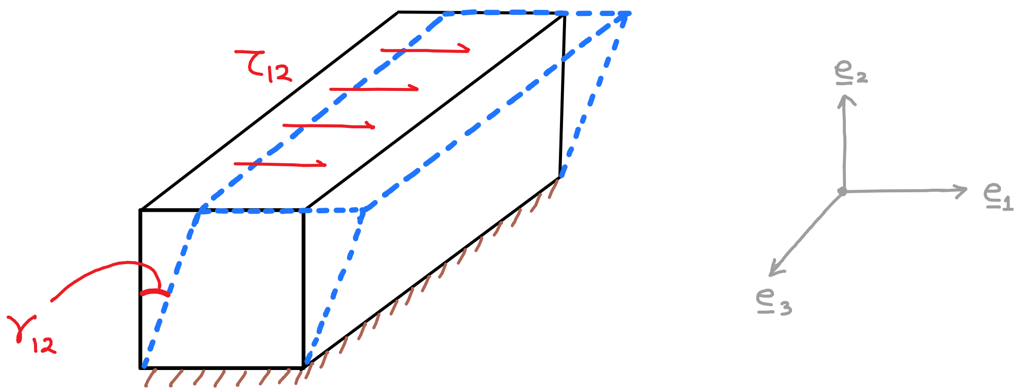
$$\gamma_{23} = 2\epsilon_{23} = \frac{\tau_{23}}{G}$$

We can see that shear modulus is given by

$$G = \frac{\tau_{12}}{\gamma_{12}} = \frac{\tau_{13}}{\gamma_{13}} = \frac{\tau_{23}}{\gamma_{23}}$$

So if we induce shear in a body and measure the corresponding shear stress, the ratio of stress to strain will give us the shear modulus.

Motivated by this, let's do an experiment with the rectangular bar, where the bar is fixed at the bottom and we apply shear force on the top face (with normal $\underline{e}_1$)



This effectively imposes shear stress τ_{12} in the body. Due to this, the bar shears : the initially perpendicular edges of the front face get inclined and shear strain γ_{12} would be equal to $\left(\frac{\pi}{2} - \text{angle between two edges}\right)$. If we draw a τ_{12} vs γ_{12} curve, the initial slope of the curve gives $G = \left.\frac{\tau_{12}}{\gamma_{12}}\right|_{\gamma_{12}=0}$

Bulk Modulus of Elasticity (K)

Young's modulus of elasticity relates normal stress with normal strain. Shear modulus relates shear stress with shear strain. We will now define bulk modulus of elasticity which relates volumetric strain (ϵ_v) with volumetric stress or equivalent pressure. When we apply pressure to a fluid (liquid/gas), its volume decreases. This decrease can be quantified by volumetric strain given by

$$\text{Volumetric strain}, \boxed{\epsilon_v = \frac{\Delta V}{V}}$$

If a pressure increase of ΔP generates volumetric strain in a fluid, bulk modulus is then given by

$$K_{fluid} = -\frac{\Delta P}{\Delta V / V}$$

For solids, the equivalent pressure is obtained from the hydrostatic part of the stress tensor, P_{eq}

$$P_{eq} = -\frac{I_1(\underline{\sigma})}{3}$$

The negative sign comes because pressure is compressive in nature while the normal component of traction is tensile when positive.

Therefore,

$$\boxed{K_{solids} = -\frac{P_{eq}}{\epsilon_v} = -\left(\frac{-I_1(\underline{\sigma})/3}{\epsilon_v}\right)}$$

That is,

$$K_{solids} = -\left(\frac{-I_1(\underline{\underline{\sigma}})/3}{\epsilon_v}\right) = \frac{I_1(\underline{\underline{\sigma}})/3}{tr(\underline{\underline{\epsilon}})} = \frac{1}{3}\,\frac{tr(\underline{\underline{\sigma}})}{tr(\underline{\underline{\epsilon}})}$$

Bulk modulus of elasticity

The bulk modulus of elasticity can also be expressed in terms of elastic constants E and ν, for isotropic material.

We have the following:

$$\epsilon_{11} + \epsilon_{22} + \epsilon_{33} = \frac{1}{E}\left(\sigma_{11} + \sigma_{22} + \sigma_{33}\right) - \frac{2\nu}{E}\left(\sigma_{11} + \sigma_{22} + \sigma_{33}\right)$$

$$\Rightarrow \quad tr(\underline{\underline{\epsilon}}) = \frac{1-2\nu}{E}\, tr(\underline{\underline{\sigma}})$$

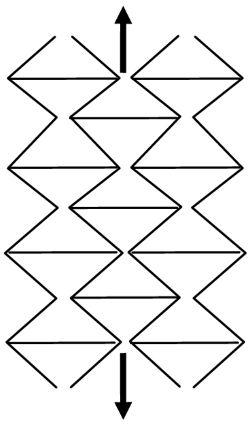
$$\Rightarrow \quad \boxed{K_{solids} = \frac{E}{3(1-2\nu)}}$$

This is an important relation. It tells us that bulk modulus K is not an independent constant. If you know the Young's modulus E and the Poisson's ratio ν, you can get the Bulk modulus using the above relation. Furthermore, this relation also gives an upper limit for the Poisson's ratio. (discussed next)

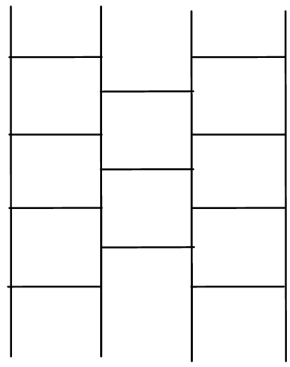
Theoretical limits for Poisson's ratio

The Poisson's ratio is usually positive. There is however a lot of research in recent years on developing materials having negative Poisson's ratio. Such materials are called auxetic materials.

Auxetic material



(a) Before loading



(b) After loading

From the relation,

$$K_{solids} = \frac{E}{3(1-2\nu)} > 0$$

we can deduce that when ν is very close to $\frac{1}{2}$, the bulk modulus becomes very large. For such materials, if we apply a finite amount of change in equivalent pressure, the volumetric strain induced in the body would be very small. This signifies **incompressibility**. Thus, $\nu \to \frac{1}{2}$ corresponds to the incompressible limit. At other extreme, materials such as cork can have Poisson's ratio close to zero.

To obtain the lower limit for the Poisson's ratio, we can use the relation

$$G = \frac{E}{2(1+\nu)} > 0$$

The Young's modulus E and the shear modulus G are both positive quantities. As a result, the denominator in the RHS must also be positive, i.e.

$$2(1+\nu) > 0 \quad \Rightarrow \quad \nu > -1$$

Thus, the theoretical limit for Poisson's ratio:

$$\boxed{-1 < \nu \leq 1/2}$$

Thermal Strain in linear elastic isotropic material

In the elastic region, the effect of temperature on strain appears in two ways:

(a) by causing modification of the elastic constants. Usually the change for most materials is very small and hence is not considered

(b) by directly producing strain in the absence of stress. The strain due to temperature change in the absence of stress is called thermal strain and is denoted by $\underset{=}{\epsilon}^{t}$

For an unconstrained isotropic material, thermal strain produces pure expansion or contraction with no shear-strain components. Also, for temp. changes of one or two hundred degrees Fahrenheit, one can closely describe the actual variation by a linear approximation.

The thermal strains due to a change in temperature from T_0 to T

$$\epsilon_{11}^{t} = \epsilon_{22}^{t} = \epsilon_{33}^{t} = \alpha\,\underbrace{(T - T_0)}_{\Delta T}$$

coefficient of thermal expansion

$$\gamma_{12}^{t} = \gamma_{23}^{t} = \gamma_{13}^{t} = 0$$

The total strain at a point in an elastic body is the sum of that due to stress and that due to temperature.

$$\underset{=}{\epsilon} = \underset{=}{\epsilon}^{e} + \underset{=}{\epsilon}^{t}$$

total strain

elastic strain

thermal strain

Thus, the stress-strain-temperature relations for a linear elastic isotropic material is as follows:

$$\epsilon_{11} = \epsilon_{11}^{e} + \epsilon_{11}^{t} = \frac{1}{E}\left(\sigma_{11} - \nu\left(\sigma_{22}+\sigma_{33}\right)\right) + \alpha \Delta T$$

$$\epsilon_{22} = \epsilon_{22}^{e} + \epsilon_{22}^{t} = \frac{1}{E}\left(\sigma_{22} - \nu\left(\sigma_{11}+\sigma_{33}\right)\right) + \alpha \Delta T$$

$$\epsilon_{33} = \epsilon_{33}^{e} + \epsilon_{33}^{t} = \frac{1}{E}\left(\sigma_{33} - \nu\left(\sigma_{11}+\sigma_{22}\right)\right) + \alpha \Delta T$$

$$\epsilon_{12} = \frac{1+\nu}{E}\tau_{12}, \qquad \epsilon_{13} = \frac{1+\nu}{E}\tau_{13}, \qquad \epsilon_{23} = \frac{1+\nu}{E}\tau_{23}$$

Alternatively, one can invert the relations and express stresses in terms of strain and temperature:

$$\sigma_{11} = \lambda\, tr(\underline{\underline{\epsilon}}) + 2\mu\,\epsilon_{11} - \beta\,\Delta T$$

$$\sigma_{22} = \lambda\, tr(\underline{\underline{\epsilon}}) + 2\mu\,\epsilon_{22} - \beta\,\Delta T$$

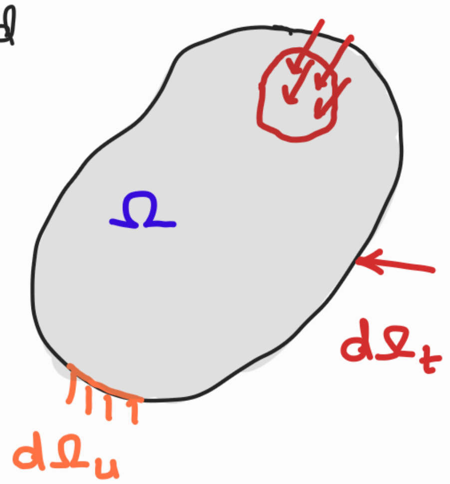
$$\sigma_{33} = \lambda\, tr(\underline{\underline{\epsilon}}) + 2\mu\,\epsilon_{33} - \beta\,\Delta T$$

$$\tau_{12} = 2\mu\,\epsilon_{12}, \qquad \tau_{13} = 2\mu\,\epsilon_{13}, \qquad \tau_{23} = 2\mu\,\epsilon_{23}$$

These provide 6 stress-strain-temperature relations.

Complete equations of linear elasticity

To know the distributions of stress and strain inside a material body Ω subject to:



a) known displacement on disp. boundary $d\Omega_u$

$$\underline{u}(\underline{x}) = \underline{u}^{\circ}(\underline{x}) \quad \text{on} \quad d\Omega_u$$

b) known forces on traction boundary $d\Omega_t$

$$\underline{\underline{\sigma}}(\underline{x})\, \underline{n}(\underline{x}) = \underline{t}^{n}(\underline{x}) \quad \text{on} \quad d\Omega_t$$

c) known increase in temperature ΔT

We can solve all 15 equations of linear elasticity

$\forall\, \underline{x}$ 3 eqns : $$\frac{\partial \sigma_{ij}}{\partial x_j} + b_i = 0$$

$\forall\, \underline{x}$ 6 $\underline{\underline{\epsilon}}-\underline{u}$ rels : $$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$\forall\, \underline{x}$ 6 $\underline{\underline{\sigma}}-\underline{\underline{\epsilon}}$ rels : $$\sigma_{ij} = \lambda\, \mathrm{tr}(\underline{\underline{\epsilon}}) + 2\mu\, \epsilon_{ij}\, \delta_{ij} - \beta \Delta T$$

B.C. : for $\underline{x}$ on $\partial\Omega_t$: $$\sigma_{ij}(\underline{x})\, n_j(\underline{x}) = t_i^{n}(\underline{x})$$

B.C. : for $\underline{x}$ on $\partial\Omega_u$: $$u_i(\underline{x}) = u_i^{\circ}(\underline{x})$$