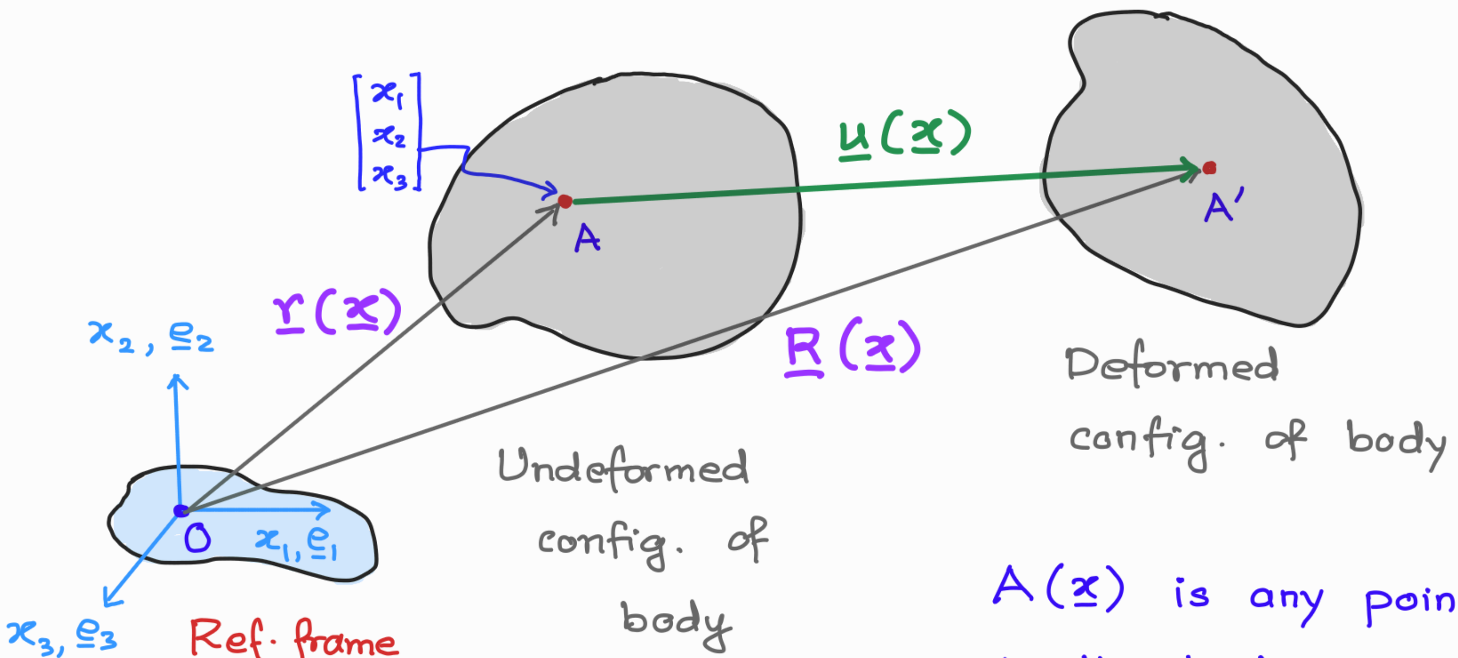
Strain - Displacement Relation

Earlier we have described normal strain and shear strain locally at a point in a body. However, even knowing these strains locally everywhere in a body is not convenient to directly determine what shape the body will actually take in its deformed configuration.

A more convenient way to define the deformation is to describe the displacement vector, $\underline{u}(\underline{x})$, for every point $A(\underline{x})$ of the body in the undeformed config. Obviously the strains are going to be related to these displacements so we need to determine these strain-displacement relations.

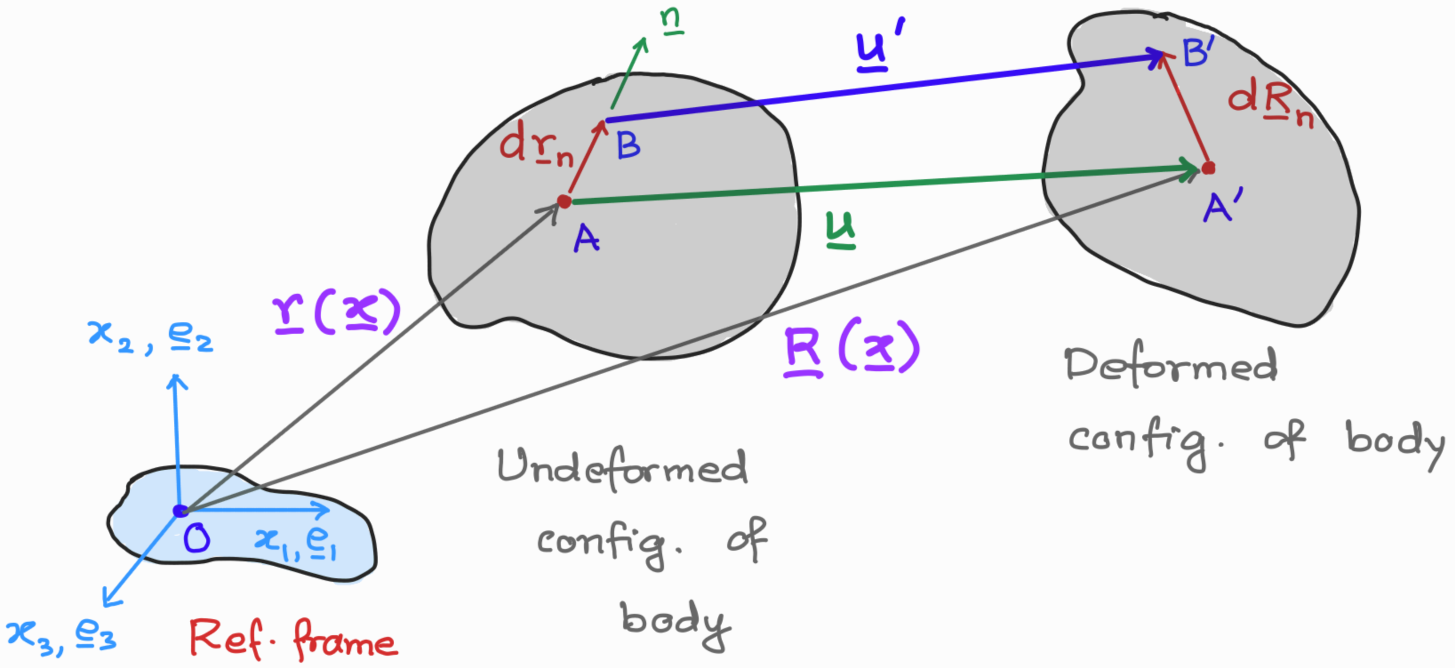


$A(\underline{x})$ is any point in the body

From the geometry we see that the position vectors of point $P(\underline{x})$ and $P^*(\underline{x})$ are related as:

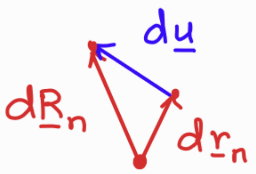
$$\underline{R}(\underline{x}) = \underline{r}(\underline{x}) + \underline{u}(\underline{x})$$

Now consider a small line segment at pt A, $d\underline{r}_n$, of length ds_n, oriented along unit normal $\underline{n}$ in the undeformed config. This line segment upon deformation becomes $d\underline{R}_n$ (of length dS_n)



$(d\underline{u} = \underline{u}' - \underline{u})$

then, we can write:



$$dR_n = d\underline{r}_n + d\underline{u}$$

Dividing the relation by the length of $d\underline{r}_n$ which is ds_n we get:

$$\frac{d\underline{R}_n}{ds_n} = \frac{d\underline{r}_n}{ds_n} + \frac{d\underline{u}}{ds_n}$$

$$= \underline{n} + \frac{d\underline{u}}{ds_n}$$

Using this result in the defn of Green-Lagrange normal strain:

$$E_{nn} = \frac{1}{2}\left(\frac{d\underline{R}_n}{ds_n}\cdot\frac{d\underline{R}_n}{ds_n} - 1\right)$$

$$= \frac{1}{2}\left[\left(\underline{n} + \frac{d\underline{u}}{ds_n}\right)\cdot\left(\underline{n} + \frac{d\underline{u}}{ds_n}\right) - 1\right]$$

$$= \frac{1}{2}\left[\overset{\cancel{1}}{\underline{n}\cdot\underline{n}} + 2\,\underline{n}\cdot\frac{d\underline{u}}{ds_n} + \frac{d\underline{u}}{ds_n}\cdot\frac{d\underline{u}}{ds_n} - \cancel{1}\right]$$

$$= \underline{n}\cdot\frac{d\underline{u}}{ds_n} + \frac{1}{2}\,\frac{d\underline{u}}{ds_n}\cdot\boxed{\frac{d\underline{u}}{ds_n}}$$

displacement gradient

Similarly, if we consider a line element $d\underline{r}_t$ of length ds_t and oriented along another unit normal $\underline{t}$, we would get E_{tt} as:

$$E_{tt} = \underline{t}\cdot\frac{d\underline{u}}{ds_t} + \frac{1}{2}\,\frac{d\underline{u}}{ds_t}\cdot\frac{d\underline{u}}{ds_t}$$

Further using the definition of Green-Lagrange shear strain:

$$E_{nt} = \frac{1}{2} \frac{dR_n}{ds_n} \cdot \frac{d\underline{R}_t}{ds_t}$$

$$= \frac{1}{2} \left[\left(\underline{n} + \frac{d\underline{u}}{ds_n} \right) \cdot \left(\underline{t} + \frac{d\underline{u}}{ds_t} \right) \right]$$

$$= \frac{1}{2} \left[\underline{t} \cdot \frac{d\underline{u}}{ds_n} + \underline{n} \cdot \frac{d\underline{u}}{ds_t} + \boxed{\frac{d\underline{u}}{ds_n} \cdot \frac{d\underline{u}}{ds_t}} \right]$$

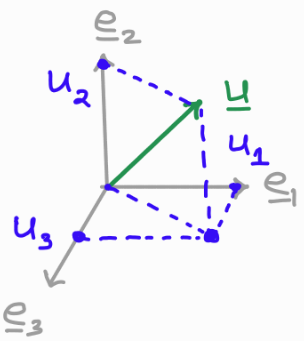
If we now consider that strains are very small s.t. we can neglect the products of the displacement gradients then we end up with infinitesimally small strains in terms of displacements at the point of interest:

$$\epsilon_{nn} = \underline{n} \cdot \frac{d\underline{u}}{ds_n}, \qquad \epsilon_{nt} = \frac{1}{2} \left(\underline{n} \cdot \frac{d\underline{u}}{ds_t} + \underline{t} \cdot \frac{d\underline{u}}{ds_n} \right)$$

The displacement vector $\underline{u}$ can be written in terms of its scalar components in the $\underline{e}_1$-$\underline{e}_2$-$\underline{e}_3$ csys.

$$\underline{u} = u_1 \underline{e}_1 + u_2 \underline{e}_2 + u_3 \underline{e}_3$$

$$= \sum_{i=1}^{3} u_i \underline{e}_i$$



The gradient of the displacement vector can then be written as:

$$\underset{\text{grad}}{\underline{\nabla}}\, \underline{u} = \frac{\partial \underline{u}}{\partial \underline{x}} = \frac{\partial \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}}{\partial \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \boxed{\begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix}}$$

often denoted by Greek letter "nabla"

in a $\underline{e}_1$-$\underline{e}_2$-$\underline{e}_3$ csys

Note the defns of normal strain ϵ_{nn} and shear strain ϵ_{nt} are valid for any mutually perpendicular directions $\underline{n}$ and $\underline{t}$

We can compute infinitesimal small strains for line segments oriented along $\underline{e}_1$, $\underline{e}_2$, and $\underline{e}_3$ directions

Normal strains along $\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$ directions

$$\epsilon_{nn} = \underline{e}_n \cdot \frac{\partial \underline{u}}{\partial x_n}$$

$\underline{n} = \underline{e}_1$ $\qquad \epsilon_{11} = \underline{e}_1 \cdot \frac{\partial \underline{u}}{\partial x_1} = \frac{\partial u_1}{\partial x_1}$

$\underline{n} = \underline{e}_2$ $\qquad \epsilon_{22} = \underline{e}_2 \cdot \frac{\partial \underline{u}}{\partial x_2} = \frac{\partial u_1}{\partial x_1}$

$\underline{n} = \underline{e}_3$ $\qquad \epsilon_{33} = \underline{e}_3 \cdot \frac{\partial \underline{u}}{\partial x_3} = \frac{\partial u_3}{\partial x_3}$

Shear strains for two mutually perpendicular line elements along $(\underline{e}_1, \underline{e}_2)$, $(\underline{e}_2, \underline{e}_3)$, $(\underline{e}_1, \underline{e}_3)$

$$\epsilon_{nt} = \frac{1}{2}\left(\underline{n} \cdot \frac{\partial \underline{u}}{\partial x_t} + \underline{t} \cdot \frac{\partial \underline{u}}{\partial x_n}\right)$$

$\underline{n} = \underline{e}_1$

$\underline{t} = \underline{e}_2$

$$\Rightarrow \quad \epsilon_{12} = \epsilon_{21} = \frac{1}{2}\left(\underline{e}_1 \cdot \frac{\partial \underline{u}}{\partial x_2} + \underline{e}_2 \cdot \frac{\partial \underline{u}}{\partial x_1}\right)$$

$$= \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$$

$\underline{n} = \underline{e}_2$

$\underline{t} = \underline{e}_3$

$$\Rightarrow \quad \epsilon_{23} = \epsilon_{32} = \frac{1}{2}\left(\underline{e}_2 \cdot \frac{\partial \underline{u}}{\partial x_3} + \underline{e}_3 \cdot \frac{\partial \underline{u}}{\partial x_2}\right)$$

$$= \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right)$$

$\underline{n} = \underline{e}_1$

$\underline{t} = \underline{e}_3$

$$\Rightarrow \quad \epsilon_{13} = \epsilon_{31} = \frac{1}{2}\left(\underline{e}_3 \cdot \frac{\partial \underline{u}}{\partial x_1} + \underline{e}_1 \cdot \frac{\partial \underline{u}}{\partial x_3}\right)$$

$$= \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right)$$

We can write the strain TENSOR using displacement gradient as follows:

2nd-order tensor $\rightarrow$
(also symmetric)

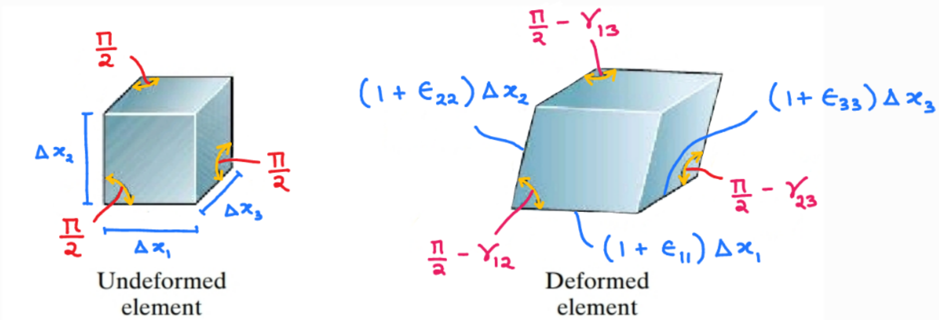
$$\boxed{\underline{\underline{\epsilon}} = \frac{1}{2}\left(\nabla \underline{u} + \nabla \underline{u}^T\right)}$$

The state of strain at a point is given by the strain tensor. The strain tensor (much like the stress tensor) depends only on the pt & not on the orientation of the plane at the pt, however, the strain matrix — which is the representation of the strain tensor in a chosen coordinate system — depends on the choice of the csys.

$$
[\epsilon]_{(e_1, e_2, e_3)} = \begin{bmatrix} \dfrac{\partial u_1}{\partial x_1} & \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial u_2}{\partial x_1}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial x_3} + \dfrac{\partial u_3}{\partial x_1}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial u_2}{\partial x_1}\right) & \dfrac{\partial u_2}{\partial x_2} & \dfrac{1}{2}\left(\dfrac{\partial u_2}{\partial x_3} + \dfrac{\partial u_3}{\partial x_2}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial x_3} + \dfrac{\partial u_3}{\partial x_1}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_2}{\partial x_3} + \dfrac{\partial u_3}{\partial x_2}\right) & \dfrac{\partial u_3}{\partial x_3} \end{bmatrix}
$$

$$
\begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{12} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{13} & \epsilon_{23} & \epsilon_{33} \end{bmatrix}
$$

If the strain tensor at a pt is known, one can describe the deformation of a small cuboidal element at that pt whose face normals are oriented along the coor. axes



Undeformed element

Deformed element