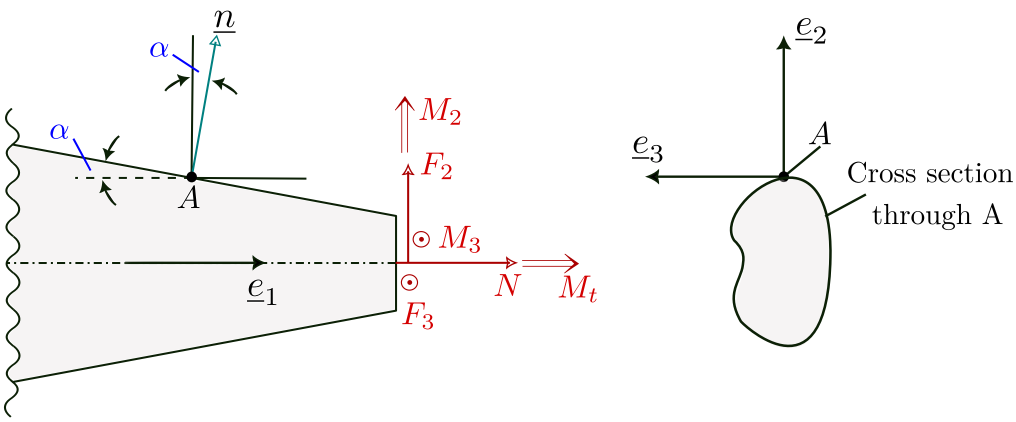
Total marks: 20 **Total time**: 2 hours

Answers without clear motivation/reasoning and handwriting will get zero marks!

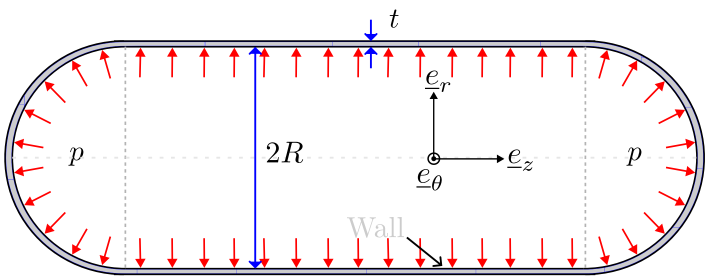
1. **[2 marks]** At point A of a beam (shown below), the components of the stress tensor expressed in $\underline{e}_1 - \underline{e}_2 - \underline{e}_3$ CSYS are as follows: σ_{11}, τ_{21}, τ_{31} are non-zero and $\sigma_{22} = \sigma_{33} = \tau_{32} = 0$. $\underline{n}$ is the normal to the lateral surface at A with $\alpha \ll 1$.



Find the error in the satisfaction of the traction-free (i.e., zero traction) boundary condition at A for the beam subjected to bending moment, axial force, and shear force as shown in the diagram.

2. **[4 marks]** Consider a closed-end **thin-walled** cylindrical shell (like a capsule) with internal radius R and thickness t subjected to a uniform internal pressure p. Show that the principal stresses in the **cylinder wall** (not the end cap walls), in cylindrical coordinates ($\underline{e}_r - \underline{e}_\theta - \underline{e}_z$), are given by:

$$\sigma_{rr} \approx 0, \quad \sigma_{\theta\theta} = \frac{pR}{t}, \quad \sigma_{zz} = \frac{pR}{2t}$$



What is the maximum shear stress in the $\theta - z$ plane? Is it equal to the absolute maximum shear stress?

3. [**4 marks**] The deviatoric stress state is used in theories of failure since it is a measure of those stresses that produce only distortional (and no volume) changes in a body. Consider the state of stress (in MPa):

$$\left[\underline{\underline{\sigma}}\right] = \begin{bmatrix} 55 & 0 & 24 \\ 0 & 46 & 0 \\ 24 & 0 & 43 \end{bmatrix}$$

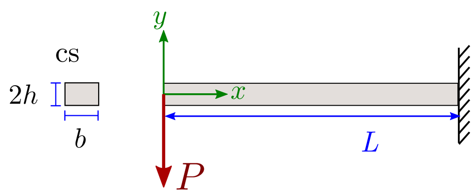
   (a) Determine the deviatoric stress matrix, $\left[\underline{\underline{\sigma}}_{\text{dev}}\right]$. **(1 mark)**

   (b) Compute the stress invariants of the stress tensor $\underline{\underline{\sigma}}$ and those of deviatoric stress tensor $\underline{\underline{\sigma}}_{\text{dev}}$. **(1 mark)**

   (c) Compute the principal directions of the deviatoric stress tensor $\left[\underline{\underline{\sigma}}_{\text{dev}}\right]$ and the stress tensor $\left[\underline{\underline{\sigma}}\right]$. How are they related?
   How are the *absolute* maximum shear stresses related in both cases? **(2 marks)**

4. [**4 marks**] Consider a cantilever beam (with one end free and another end fixed) with a vertical load P applied to the free end of the beam. Take origin at the free end. At the fixed end, the displacements and the slope are zero, i.e., $u_x(L,0) = u_y(L,0) = \frac{\partial u_y}{\partial x}(L,0) = 0$. The corresponding strain field at any point in the beam is given by:

$$\epsilon_{xx} = \frac{Pxy}{EI}, \qquad \epsilon_{yy} = -\nu\,\epsilon_{xx},$$
$$\epsilon_{xz} = \epsilon_{yz} = \epsilon_{zz} = 0,$$
$$\epsilon_{xy} = \frac{P}{4GI}(h^2 - y^2)$$



ν, E, I, G, L, h, b, P are known constants.
Check if the strain field is compatible or not. Using the given information, find out the displacement fields $u_x(x,y)$ and $u_y(x,y)$.

5. [**3 marks**] A thin rectangular sheet is strained with uniform strain components: $\epsilon_{xx} = 0.001$, $\epsilon_{yy} = -0.001$, $\gamma_{xy} = 0.001$. All other strain components are zero implying it is a plane strain condition.

   (a) Draw Mohr's circle for this case. **(1 mark)**

   (b) How much will be the percentage change in the area of the thin sheet? **(0.5 mark)**

   (c) Which line elements undergo maximum and minimum normal strain? **(1 mark)**

   (d) What will be the orientation of the two perpendicular lines in the plane of the sheet that undergo maximum angle change between them? **(0.5 mark)**

6. [**3 marks**] Show that $\underline{T}^n = \sum_i^3 \underline{T}^i\,(\underline{n}\cdot\underline{e}_i) = \sum_i^3 \underline{T}^{\hat{i}}\,(\underline{n}\cdot\hat{\underline{e}}_i)$, i.e, the formula is independent of what three planes are chosen to determine $\underline{T}^n$!

Q1) $$[\underline{n}]_{(e_1\ e_2\ e_3)} = \begin{bmatrix} \sin\alpha \\ \cos\alpha \\ 0 \end{bmatrix}$$ 1/2

(assuming a unit normal)

Since the surface is assumed to be traction-free (i.e. zero traction boundary), it must satisfy

$$\underline{\underline{\sigma}}\ \underline{n} = \underline{0}$$ 1/2

However, for the given components of $[\underline{\underline{\sigma}}]_{(e_1\ e_2\ e_3)}$, it does not satisfy zero traction.

1/2

$$\begin{bmatrix} \sigma_{11} & \tau_{21} & \tau_{31} \\ \tau_{21} & \cancelto{0}{\sigma_{22}} & \cancelto{0}{\tau_{32}} \\ \tau_{31} & \cancelto{0}{\tau_{32}} & \cancelto{0}{\sigma_{33}} \end{bmatrix} \begin{bmatrix} \sin\alpha \\ \cos\alpha \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The errors are

$$\text{error } 1 = \sigma_{11}\sin\alpha + \tau_{21}\cos\alpha$$

$$\text{error } 2 = \tau_{21}\sin\alpha$$

$$\text{error } 3 = \tau_{31}\sin\alpha$$

1/2

2) Generally, the state of stress can vary from point to point.

Therefore, $\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{zz}, \tau_{z\theta}, \tau_{r\theta}, \tau_{rz} \rightarrow f(r, \theta, z)$

- Due to symmetry about z-axis, there would be no variation of any stresses along z-direction and θ-direction

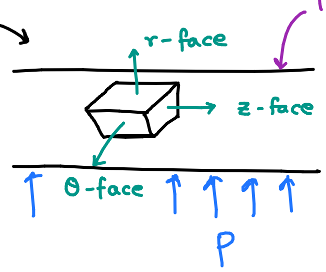
$$\therefore \quad \sigma_{rr}, \sigma_{\theta\theta}, \sigma_{zz}, \tau_{z\theta}, \tau_{r\theta}, \tau_{rz} \rightarrow f(r)$$ 1/2

- $\sigma_{rr} = \tau_{rz}$ = constant (due to negligible variation across in cylindrical wall)

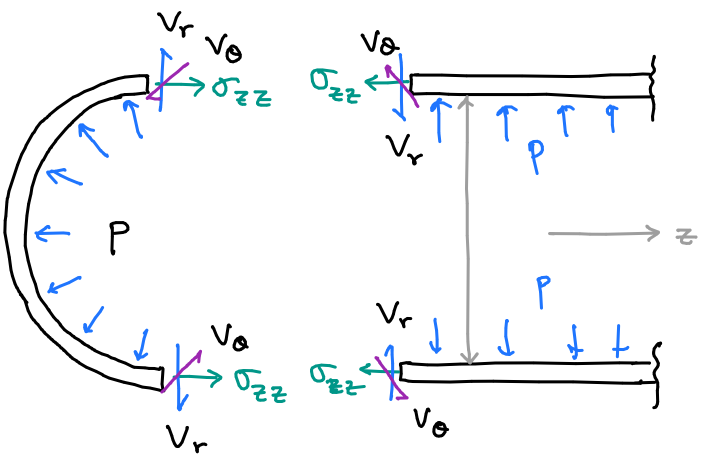
But, $\sigma_{rr}\big|_{\text{inner wall}} = \tau_{rz} = \sigma_{rr}\big|_{\text{outer wall}}$ = constant

3/4

$$\therefore \quad \sigma_{rr}\big|_{\text{outer face}} = 0$$
$$\sigma_{rr}\big|_{\text{inner face}} = -P$$
$$\tau_{rz} = 0$$



- At the joint of the end caps to the cylindrical wall,



Using force equilibrium in z-direction:

$$\overset{+}{\rightarrow} \sum F_z = 0$$

$$\Rightarrow \sigma_{zz}(2\pi r t) = p\pi r^2$$ 1/2

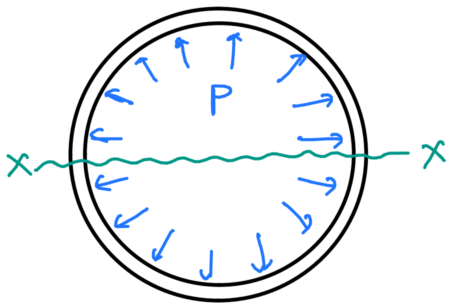
$$\Rightarrow \sigma_{zz} = \frac{pr}{2t}$$

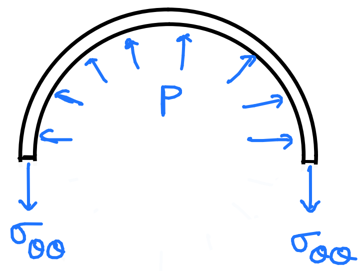
1/2 $V_r = 0$ (else one side should bulge out and other side should bulge in)

$$\therefore \tau_{z\theta} = 0$$

(since there is no twisting moment about z) 1/2

Now looking at a cross-section of cylinder wall





$$+\uparrow \sum F_y = 0 \quad \Rightarrow \quad 2\,\sigma_{\theta\theta}(t)\cdot(1) = p\,(2r)\,(1)$$

$$\Rightarrow \quad \sigma_{\theta\theta} = \frac{pr}{t}$$

1/2

There would be no shear, so $\tau_{r\theta} = 0$ 1/4

Maximum shear stress in θ-z plane $\rightarrow$ $\left|\dfrac{\gamma_\theta - \gamma_z}{2}\right|$

$$= \left|\frac{\sigma_{\theta\theta} - \sigma_{zz}}{2}\right|$$

$$= \frac{\frac{pr}{t} - \frac{pr}{2t}}{2}$$

$$= \frac{pr}{4t}$$ 1/2

Absolute max. shear stress $\rightarrow$ $\left|\dfrac{\gamma_\theta - \gamma_r}{2}\right| = \left|\dfrac{\sigma_{\theta\theta} - \sigma_{rr}}{2}\right|$

$$= \frac{pr}{2t}$$ 1/2

No they are not equal!

3) (a)

$$[\underline{\underline{\sigma}}_{dev}] = \begin{bmatrix} 55 & 0 & 24 \\ 0 & 46 & 0 \\ 24 & 0 & 43 \end{bmatrix} - 48 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$48 = \frac{55+46+43}{3}$$

$$= \begin{bmatrix} 7 & 0 & 24 \\ 0 & -2 & 0 \\ 24 & 0 & -5 \end{bmatrix}$$ 1

(b)

$$I_1(\sigma) = 144 \quad 1/8$$

$$I_2(\sigma) = \begin{vmatrix} 55 & 0 \\ 0 & 46 \end{vmatrix} + \begin{vmatrix} 46 & 0 \\ 0 & 43 \end{vmatrix} + \begin{vmatrix} 55 & 24 \\ 24 & 43 \end{vmatrix}$$

$$= 2530 + 1978 + 1789$$

$$= 6297 \quad 2/8$$

$$I_3(\sigma) = \det([\underline{\underline{\sigma}}]) = 82294 \quad 1/8$$

1/2

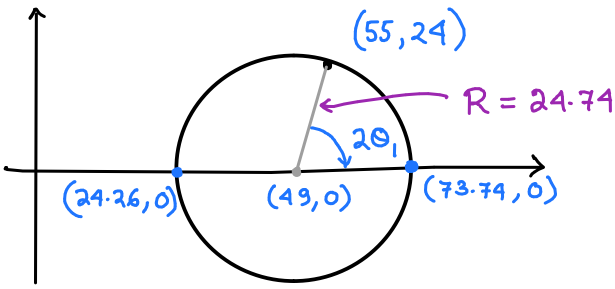
$$I_1(\sigma_{dev}) = 0 \quad 1/8$$

$$I_2(\sigma_{dev}) = -615 \quad 2/8$$

$$I_3(\sigma_{dev}) = 1222 \quad 1/8$$

1/2

(c)



We can draw Mohr's circle for

$$[\underline{\underline{\sigma}}] = \begin{bmatrix} 55 & 24 \\ 24 & 43 \end{bmatrix}$$

State of stress $[\underline{\underline{\sigma}}]$

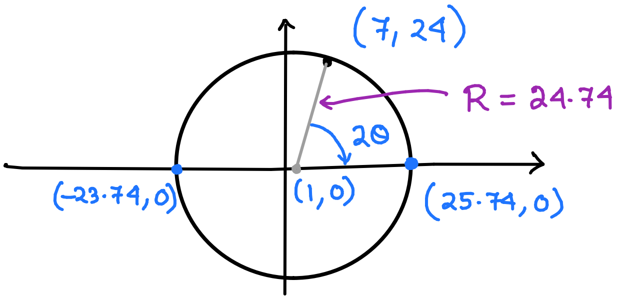
Principal stresses of deviatoric state = 73.74 , 46 , 24.26

Angles : $\theta_1 = \frac{1}{2} \tan^{-1}\left(\frac{24}{6}\right)$ anticlockwise from x-axis

$\theta_2 = \frac{\pi}{2} + \theta_1$

θ_3 = along y-axis

1/2



We can draw Mohr's circle for

$$[\underline{\sigma}_{dev}] = \begin{bmatrix} 7 & 24 \\ 24 & -5 \end{bmatrix}$$

Deviatoric

Principal stresses of deviatoric state $= 25.74, \ -2, \ -23.74$

Angles : $\theta_1 = \frac{1}{2} \tan^{-1}\left(\frac{24}{6}\right)$ anticlockwise from x-axis

$\theta_2 = \frac{\pi}{2} + \theta_1$

θ = along y-axis

1/2

Principal directions remain the same!

Alternatively,

$$[\underline{n}_1] = \begin{bmatrix} \cos\theta_1 \\ 0 \\ \sin\theta_1 \end{bmatrix}, \quad [\underline{n}_2] = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad [\underline{n}_3] = \begin{bmatrix} \cos\theta_2 \\ 0 \\ \sin\theta_2 \end{bmatrix}$$

(e) Principal stresses for $[\underline{\sigma}] \rightarrow 73.74, \ 46, \ 24.26$

Absolute max shear stress $= \dfrac{73.74 - 24.26}{2}$

$= 24.74$

1/2

Principal stresses of deviatoric state $= 25.74, \ -2, \ -23.74$

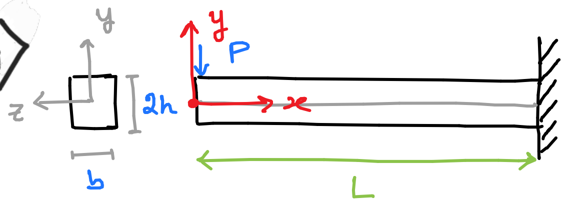
Absolute max shear stress $= \dfrac{25.74 - (-23.74)}{2}$

$= 24.74$

1/2

They are same!

4)



Given:

$$\epsilon_{xx} = \frac{Pxy}{EI}$$

$$\epsilon_{yy} = -\nu\,\epsilon_{xx}$$

$$\epsilon_{xz} = \epsilon_{yz} = \epsilon_{zz} = 0$$

$$\epsilon_{xy} = \frac{P}{4GI}(h^2-y^2)$$

$$u_x\big|_{(L,0)} = 0$$

$$u_y\big|_{(L,0)} = 0$$

$$\frac{\partial u_y}{\partial x}\bigg|_{(L,0)} = 0$$

Integrate the normal strains ϵ_{xx} and ϵ_{yy}

$$\frac{\partial u_x}{\partial x} = \epsilon_{xx} \Rightarrow u_x(x,y) = \frac{Px^2y}{2EI} + g(y) \quad \text{--- (1)}$$ 1/2

$$\frac{\partial u_y}{\partial y} = -\nu\,\epsilon_{xx} \Rightarrow u_y(x,y) = -\nu\frac{Pxy^2}{2EI} + h(x) \quad \text{--- (2)}$$ 1/2

From shear strain, we find

$$\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = 2\,\epsilon_{xy} = \frac{P}{2GI}(h^2-y^2)$$

which yields using (1) and (2)

$$\frac{Px^2}{2EI} + \frac{dg}{dy} - \frac{\nu P y^2}{2EI} + \frac{dh}{dx} = \frac{P}{2GI}(h^2-y^2)$$ 1/2

Rearranging the terms, we get:

$$\underbrace{\frac{Px^2}{2EI} + \frac{dh(x)}{dx}}_{H(x)} = \underbrace{\frac{P}{2GI}(h^2-y^2) + \frac{\nu P y^2}{2EI} - \frac{dg(y)}{dy}}_{G(y)}$$

The above equation is of the form $H(x) = G(y)$. Hence, it must be that

$$H(x) = G(y) = C \quad \text{(constant)}$$ 1/2

Therefore, integrating the LHS and RHS, we get

LHS: $$\frac{Px^2}{2EI} + \frac{dh(x)}{dx} = c \Rightarrow h(x) = Cx - \frac{Px^3}{6EI} + D_1$$

RHS: $$\frac{P}{2GI}(h^2 - y^2) + \frac{\nu P y^2}{2EI} - \frac{dg(y)}{dy} = c$$

$$\Rightarrow g(y) = -Cy + \frac{P}{2GI}\left(h^2 y - \frac{y^3}{3}\right) + \frac{\nu P y^3}{6EI} + D_2$$

Putting these values in the displacement expressions, we get:

$$u_x(x,y) = \frac{Px^2 y}{2EI} + \frac{P}{2GI}\left(h^2 y - \frac{y^3}{3}\right) + \frac{\nu P y^3}{6EI} - Cy + D_2$$

$$u_y(x,y) = -\frac{Px^3}{6EI} - \frac{\nu P x y^2}{2EI} + Cx + D_1$$

Now, use the displacement boundary conditions

$$u_x\big|_{(L,0)} = 0 \Rightarrow D_2 = 0$$

$$u_y\big|_{(L,0)} = 0 \Rightarrow D_1 = -CL + \frac{PL^3}{6EI}$$

$$\frac{\partial u_y}{\partial x}\bigg|_{(L,0)} = 0 \Rightarrow -\frac{PL^2}{2EI} + C = 0 \Rightarrow C = \frac{PL^2}{2EI}$$

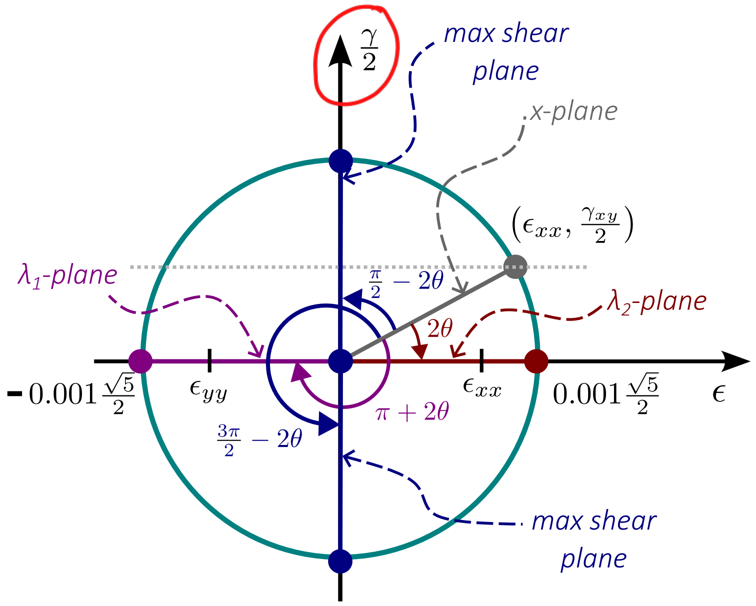
$$\therefore \quad D_1 = -\frac{PL^3}{3EI}, \quad D_2 = 0, \quad C = \frac{PL^2}{2EI} \quad \Bigg\} \quad 1.5$$

For plane strain case, five strain-compatibility equations are naturally satisfied. You will also find that

$$* \quad \frac{\partial^2 \epsilon_{11}}{\partial y^2} + \frac{\partial^2 \epsilon_{22}}{\partial x^2} = 2\frac{\partial^2 \epsilon_{12}}{\partial x \partial y}$$ is also satisfied! ½

Q5) A thin rectangular sheet is strained with uniform strain components: $\epsilon_{xx} = 0.001$, $\epsilon_{yy} = -0.001$, $\gamma_{xy} = 0.001$. All other strain components are zero implying it is a plane strain condition.
(i) Draw Mohr's circle for this case.
(ii) How much will be the percentage change in the area of thin sheet?
(iii) Which line elements undergo maximum and minimum longitudinal/normal strain?
(iv) What will be the orientation of the two perpendicular lines in the plane of sheet which undergo maximum angle change between them?

Solution:
$\epsilon_{xx} = 0.001$, $\epsilon_{yy} = 0.001$, $\gamma_{xy} = 0.001$, others are all zero!



(i) The Mohr's circle is drawn above.

$$\tan 2\theta = \frac{\gamma_{xy}/2}{(\epsilon_{xx} - \epsilon_{yy})/2} = \frac{0.0005}{0.001} = \frac{1}{2} \qquad \text{1/2}$$

$$\text{Center of the circle} = \frac{\epsilon_{xx} + \epsilon_{yy}}{2} = 0 \qquad \text{1/4}$$

$$\text{Radius} = \sqrt{\left(\frac{\epsilon_{xx} - \epsilon_{yy}}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} = \sqrt{0.001^2 + \frac{0.001^2}{4}} = 0.001\frac{\sqrt{5}}{2} \qquad \text{1/4}$$

(ii) Percentage change in area of the sheet = $(\epsilon_{xx} + \epsilon_{yy}) \times 100 = 0\%$ 1/2

(iii) To find line elements undergoing max. and min. longitudinal strain (principal strains), we use the Mohr's circle. The principal strains are:

Part marks here are calculated based on the Mohr's circle drawn (even if incorrect Mohr's circle)

- $\dfrac{\epsilon_{xx} + \epsilon_{yy}}{2} + R = R$: the line element that undergoes maximum longitudinal strain lie along the normal of maximum principal strain plane. Its normal is oriented at

$$2\theta = \tan^{-1}\left(\frac{1}{2}\right) \text{ clockwise from X-axis in Mohr's cirle}$$
$$\Rightarrow \theta = \frac{1}{2}\tan^{-1}\left(\frac{1}{2}\right) \text{ anti-clockwise from X-axis physically.}$$

1/2

- $\dfrac{\epsilon_{xx} + \epsilon_{yy}}{2} - R = -R$; the line element that undergoes minimum longitudinal strain lie along the normal of minimum principal strain plane. Its normal is oriented at

$$\pi + 2\theta = \pi + \tan^{-1}\left(\frac{1}{2}\right) \text{ clockwise from X-axis in Mohr's cirle}$$
$$\Rightarrow \frac{\pi}{2} + \theta = \frac{\pi}{2} + \frac{1}{2}\tan^{-1}\left(\frac{1}{2}\right) \text{ anti-clockwise from X-axis physically.}$$

1/2

(iv) Maximum shear strain will occur between those two line elements which correspond to top and bottom points in the circle. The corresponding directions are:

- First line element oriented along

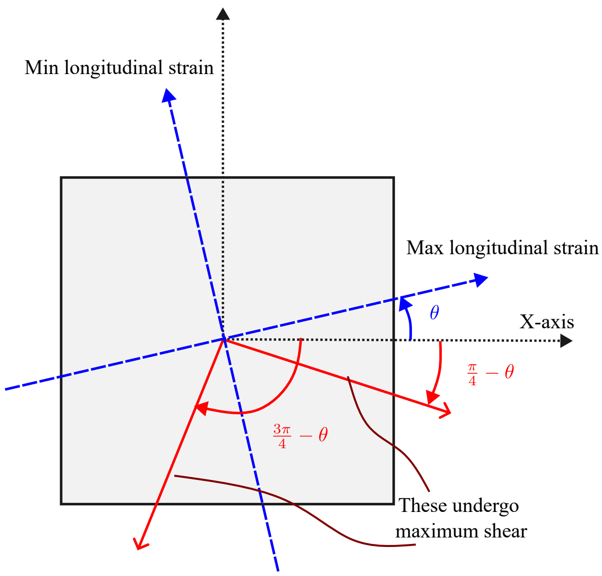
$$\left(\frac{\pi}{2} - 2\theta\right) \text{ anti-clockwise from X-axis in Mohr's circle}$$
$$\Rightarrow \left(\frac{\pi}{4} - \theta\right) \text{ clockwise from X-axis physically}$$

1/4

- Second line element oriented along

$$\left(\frac{3\pi}{2} - 2\theta\right) \text{ anti-clockwise from X-axis in Mohr's circle}$$
$$\Rightarrow \left(\frac{3\pi}{4} - \theta\right) \text{ clockwise from X-axis physically}$$

1/4



Q6) This question is from Tutorial 3

$$\underline{e}_j = \sum_i (\underline{e}_j \cdot \hat{\underline{e}}_i)\, \hat{\underline{e}}_i \qquad \Big\}\ 1$$

$$\underline{T}^n = \sum_{i=1}^{3} \underline{T}^{\hat{i}}\, (\underline{n} \cdot \hat{\underline{e}}_i)$$

Express $\underline{T}^{\hat{i}}$ in terms of traction on planes $\underline{e}_1 - \underline{e}_2 - \underline{e}_3$

$$\underline{T}^{\hat{i}} = \sum_{j=1}^{3} \underline{T}^{j}\, (\hat{\underline{e}}_i \cdot \underline{e}_j) \qquad \Big\}\ 1/2$$

Now sub the value of $\underline{T}^{\hat{i}}$ in the expression of $\underline{T}^n$

$$\begin{aligned}
\underline{T}^n &= \sum_{i=1}^{3} \underline{T}^{\hat{i}}\, (\underline{n} \cdot \hat{\underline{e}}_i) \\
&= \sum_{i=1}^{3} \left(\sum_{j=1}^{3} \underline{T}^{j}\, (\hat{\underline{e}}_i \cdot \underline{e}_j) \right) (\underline{n} \cdot \hat{\underline{e}}_i) \\
&= \sum_{i=1}^{3} \sum_{j=1}^{3} \underline{T}^{j}\ \underline{n} \cdot \left((\hat{\underline{e}}_i \cdot \underline{e}_j)\, \hat{\underline{e}}_i \right) \\
&= \sum_{j=1}^{3} \underline{T}^{j}\ \underline{n} \cdot \underbrace{\left(\sum_{i=1}^{3} (\underline{e}_j \cdot \hat{\underline{e}}_i)\, \hat{\underline{e}}_i \right)}_{= \underline{e}_j}
\end{aligned} \qquad \Bigg\}\ 1$$

$$\begin{aligned}
&= \sum_{j=1}^{3} \underline{T}^{j}\, (\underline{n} \cdot \underline{e}_j) \\
&= \sum_{i=1}^{3} \underline{T}^{i}\, (\underline{n} \cdot \underline{e}_i)
\end{aligned} \qquad \Big\}\ 1/2$$