MAJOR (35 marks, 2 hours)

1. [**3 marks**] You have learned in class the relation between stress and strain components as either

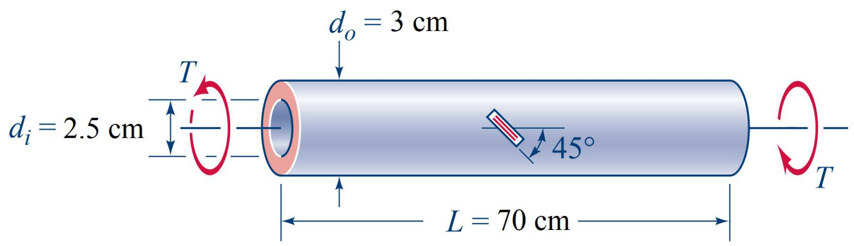
$$\sigma_{ij} = \lambda(\epsilon_{11} + \epsilon_{22} + \epsilon_{33}) + 2\mu\epsilon_{ij}, \qquad i, j = 1, 2, 3$$

or

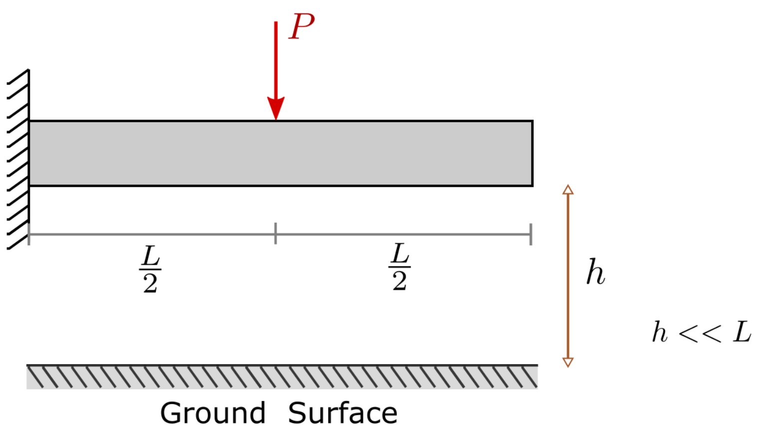
$$\epsilon_{11} = \frac{1}{E}(\sigma_{11} - \nu(\sigma_{22} + \sigma_{33})), \epsilon_{22} = \dots, \epsilon_{33} = \dots$$
$$\gamma_{12} = \frac{\tau_{12}}{G}, \gamma_{23} = \dots, \gamma_{13} = \dots$$

Now derive a simplified relation of (E, G, ν) in terms of Lame's constants (λ, μ) for a 3D bar of uniform cross-sectional area under uniaxial stress along $\underset{\sim}{e}_1$ direction.

2. [**8 marks**] The tube shown in figure, with outer diameter $d_o = 3$ cm and inner diameter $d_i = 2.5$ cm, is subjected to a torque of $T = 110$ N-m. A strain gauge oriented at an angle $\theta = -45°$ with respect to the axis, which measures the extensional strain along this direction, gives a reading of 190×10^{-6}. (a) Determine the value of the maximum shear stress, $T_{\max}$, (b) determine the shear modulus of elasticity, G, and (c) determine the angle of twist in a section of the tube of length $L = 70$ cm.

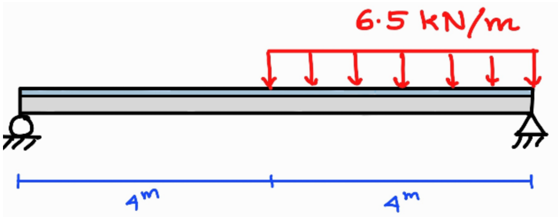


3. [**10 marks**] Think of a cantilever beam which is loaded in the middle. How much of the beam would be in contact with the surface below? Neglect the weight of the beam. Model the cantilever as an Euler-Bernoulli beam. Assume the load applied is such that a portion of the beam in contact with the ground surface is less than the beam's half length.

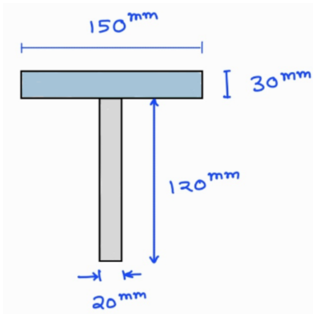


4. **[6 marks]** The beam shown in the figure below is made from two boards. Draw the shear force diagram for the beam. Show that the neutral axis (NA) of the cross-section lies 108.91 mm from the bottom of the cross-section. What is the moment of inertia I_{zz} about the NA?
Determine the maximum shear stress in the glue necessary to hold the boards together along the seam where they are joined.
Hint: Use the formula for shear stress is

$$\tau_{xy} = \frac{V(x)\,Q(y)}{I_{zz}\,b(y)}$$

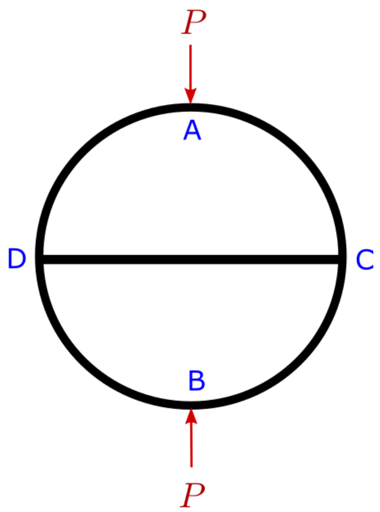


(a) Transverse loading on simply supported beam



(b) Cross-section of beam

5. **[8 marks]** In the tutorial, we learned about the deformation of a square beam subjected to equal and opposite diametrical loads. Now consider a ring beam with a straight member along the horizontal diameter as shown in the figure below. By how much will the ring contract along the line of the applied load? By how much will the straight member elongate?



Radius of ring $= R$

Cross-section radius $= r$

$(r << R)$

Q1>

Considering the case of uniaxial loading, we can see that

$$\epsilon_{11} = \frac{1}{E}\left(\sigma_{11} - \nu(\cancelto{0}{\sigma_{22}} + \cancelto{0}{\sigma_{33}})\right)$$

$$\Rightarrow \quad \sigma_{11} = E\,\epsilon_{11}$$

$$\epsilon_{22} = \frac{1}{E}\left(\cancelto{0}{\sigma_{22}} - \nu(\sigma_{11} + \cancelto{0}{\sigma_{33}})\right) = -\frac{\nu\, \cancelto{E\epsilon_{11}}{\sigma_{11}}}{E} = -\nu\,\epsilon_{11}$$

$$\epsilon_{33} = -\nu\,\epsilon_{11}$$

Now using Lame's constant, we can write

1/2

$$\sigma_{11} = \lambda(\epsilon_{11} + \epsilon_{22} + \epsilon_{33}) + 2\mu\,\epsilon_{11}$$

$$\Rightarrow E\epsilon_{11} = \lambda(1-2\nu)\,\epsilon_{11} + 2\mu\,\epsilon_{11}$$

$$\Rightarrow E = \lambda(1-2\nu) + 2\mu$$

$$\Rightarrow E = (\lambda + 2\mu) - 2\lambda\nu \quad \text{——} \; (1)$$

$$\tau_{12} = \mu\,\gamma_{12}$$

$$\tau_{12} = G\,\gamma_{12}$$

$$\therefore \quad \mu = G$$

1/2

1

$$\sigma_{22} = \lambda(\epsilon_{11} + \epsilon_{22} + \epsilon_{33}) + 2\mu\,\epsilon_{22}$$

$$\Rightarrow 0 = \lambda(1-2\nu)\,\epsilon_{11} - 2\mu\nu\,\epsilon_{11}$$

$$\Rightarrow 0 = \lambda - 2(\lambda+\mu)\,\nu$$

$$\Rightarrow \nu = \frac{\lambda}{2(\lambda+\mu)} \quad \text{——} \; (2)$$

Use (2) in (1) :

$$E = (\lambda + 2\mu) - \frac{2}{2}\,\frac{\lambda^2}{(\lambda+\mu)}$$

$$\Rightarrow E = \frac{(\lambda+2\mu)(\lambda+\mu) - \lambda^2}{(\lambda+\mu)}$$

$$= \frac{\lambda^2 + 3\lambda\mu + 2\mu^2 - \lambda^2}{(\lambda+\mu)}$$

$$= \frac{\mu(3\lambda + 2\mu)}{(\lambda+\mu)}$$

1

Q2) a) The maximum shear stress occurs at the outer surface and is given by the torsion formula:

$$\tau_{max} = \frac{T r_o}{J} \quad 1$$

where $J = \frac{\pi}{32}\left(d_o^4 - d_i^4\right) = \frac{\pi}{32}\left[(0.03)^4 - (0.025)^4\right]$

$$= 4.117 \times 10^{-8} \ m^4 \quad 1$$

Then,

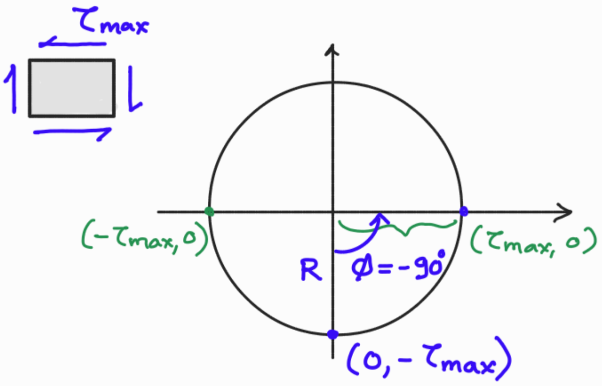
$$\tau_{max} = \frac{(110 \ N.m)(0.015 \ m)}{4.117 \times 10^{-8} \ m^4}$$

$$= 4.0075 \times 10^{6} \ N/m^2 \quad 1$$

b) From Hooke's law, $\gamma = \tau / G \Rightarrow G = \frac{\tau}{\gamma}$

however, shear strain γ is not measured in the problem

Given, the tube wall is only subjected to a pure shear state of stress, the Mohr's circle gives:

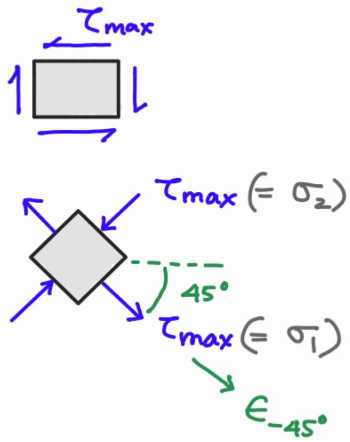


* Max/min normal stresses

$$\sigma_{1,2} = \pm \tau_{max} \quad 1$$

* $\theta = -45°$ is the direction of maximum tension 1

* By Hooke's law,



$$\epsilon_{-45°} = \frac{1}{E}\left[\sigma_1 - \nu\,\sigma_2\right]$$

$$= \frac{1}{E}\left[\tau_{max} - \nu(-\tau_{max})\right]$$

$$= \frac{\tau_{max}}{E}(1+\nu)$$

1 $\quad \boxed{\epsilon_{-45°} = \frac{\tau_{max}}{2G}}$ $\quad \left(\because \; G = \frac{E}{2(1+\nu)}\right)$

Using this relation, we get shear modulus G :

$$G = \frac{\tau_{max}}{2\epsilon_{-45°}} = \frac{4.0075 \times 10^{6} \text{ N/m}^2}{2(190 \times 10^{-6})} = \underline{1.054 \times 10^{11} \text{ N/m}^2}$$

1

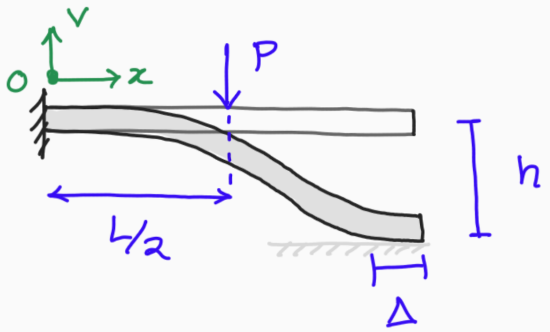
c) Angle of twist,

$$\phi = \frac{TL}{GJ}$$ | 1/2

$$= \frac{(110 \text{ N·m})(0.7 \text{ m})}{(1.054 \times 10^{11} \text{ N/m}^2)(4.117 \times 10^{-8} \text{ m}^4)}$$

$$= 0.00177 \text{ rad}$$ | 1/2

Q3>



A portion of the beam, Δ, would be resting on the surface

At this end, we will have $v(x) = -h$ for $x > L-\Delta$

$\Rightarrow$ end disp $= -h$ ① $v(L-\Delta) = -h$

$\Rightarrow$ slope $= 0$ ② $\left.\dfrac{dv}{dx}\right|_{x=L-\Delta} = 0$

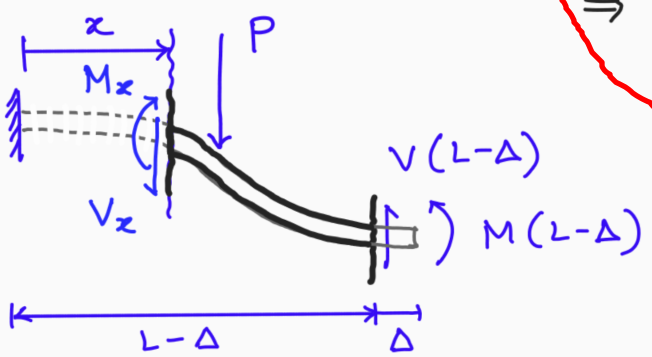
$\Rightarrow$ curvature $= 0$ ③ $\left.\dfrac{d^2v}{dx^2}\right|_{x=L-\Delta} = 0$

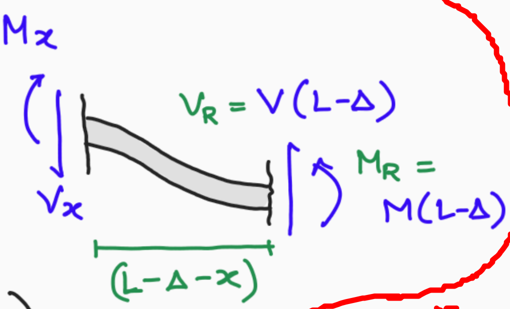
2.5

$\Rightarrow$ deflection & slope $= 0$ at left end

④ $v(x=0) = 0$

⑤ $\dfrac{dv}{dx}(x=0) = 0$

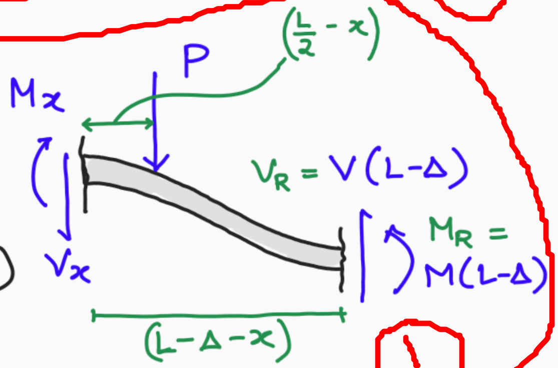




For $x > L/2$, $V(x) = V_R$

$M(x) = M_R + V_R\,(L-\Delta-x)$

1



For $x < L/2$, $V(x) = V_R - P$

$M(x) = M_R + V_R\,(L-\Delta-x) - P\left(\frac{L}{2} - x\right)$

1

By Euler-Bernoulli theory, $EI \dfrac{d^2 v(x)}{dx^2} = M(x)$ 1/2

For $0 < x < L/2$

$$EI \frac{d^2 v}{dx^2} = M_R + V_R (L - \Delta - x) - P\left(\frac{L}{2} - x\right)$$

A1

$$\Rightarrow EI \frac{dv(x)}{dx} = M_R x + V_R\left[(L-\Delta)x - \frac{x^2}{2}\right] - P\left[\frac{L}{2}x - \frac{x^2}{2}\right] + C_1$$

Using BC 5, $\left.\dfrac{dv}{dx}\right|_{x=0} = 0$, we get $C_1 = 0$ 1/2

Integrating further,

$$\Rightarrow EI\, v(x) = \frac{M_R x^2}{2} + V_R\left[\frac{(L-\Delta)x^2}{2} - \frac{x^3}{6}\right] - P\left[\frac{Lx^2}{4} - \frac{x^3}{6}\right] + C_2$$

Using BC 4, $v|_{x=0} = 0$, we get $C_2 = 0$ 1/2

$$\Rightarrow v(x) = \frac{1}{EI}\left[\frac{M_R x^2}{2} + V_R\left(\frac{(L-\Delta)x^2}{2} - \frac{x^3}{6}\right) - P\left(\frac{Lx^2}{4} - \frac{x^3}{6}\right)\right]$$

Disp at $x = L/2^-$

$$v(x = L/2^-) = \frac{1}{EI}\left[\frac{M_R L^2}{8} + V_R\left(\frac{(L-\Delta)L^2}{8} - \frac{L^3}{48}\right) - P\left(\frac{L^3}{16} - \frac{L^3}{48}\right)\right]$$

$$= \frac{1}{EI}\left[\frac{M_R L^2}{8} + \frac{5V_R L^3}{48} - \frac{V_R \Delta L^2}{8} - \frac{PL^3}{24}\right]$$

Slope at $x = L/2^-$

$$\frac{dv}{dx}\left(x = \frac{L}{2}^-\right) = M_R\frac{L}{2} + V_R\left[(L-\Delta)\frac{L}{2} - \frac{L^2}{8}\right] - P\left[\frac{L^2}{4} - \frac{L^2}{8}\right]$$

$$= \frac{M_R L}{2} + V_R\left(\frac{3L^2}{8} - \frac{\Delta L}{2}\right) - \frac{PL^2}{8}$$

V_R, M_R, Δ are the remaining three unknowns which occur at $x > L/2$

For $x > L/2$

$$EI\frac{d^2v}{dx^2} = M_R + V_R(L-\Delta-x)$$

Using BC ③, $\left.\frac{d^2v}{dx^2}\right|_{x=L-\Delta} = 0$,

$$\Rightarrow \quad 0 = M_R + V_R\left(L-\Delta-(L-\Delta)\right) \Rightarrow \boxed{M_R = 0} \quad 1$$

$$\Rightarrow EI\frac{dv}{dx} = V_R\left((L-\Delta)x - \frac{x^2}{2}\right) + b_1$$

Using BC ②, $\left.\frac{dv}{dx}\right|_{x=L-\Delta} = 0$

$$\Rightarrow \quad 0 = V_R\left[(L-\Delta)^2 - \frac{(L-\Delta)^2}{2}\right] + b_1$$

$$\Rightarrow \quad b_1 = -V_R\frac{(L-\Delta)^2}{2} \quad 1$$

$$EI\frac{dv}{dx} = V_R\left((L-\Delta)x - \frac{x^2}{2} - \frac{(L-\Delta)^2}{2}\right)$$

Integrating further,

$$EI\ v(x) = V_R\left[(L-\Delta)\frac{x^2}{2} - \frac{x^3}{6} - \frac{(L-\Delta)^2}{2}x\right] + b_2$$

Using BC 1, $v(x=L-\Delta) = -h$

$$-EIh = V_R\left[\cancel{\frac{(L-\Delta)^3}{2}} - \frac{(L-\Delta)^3}{6} - \cancel{\frac{(L-\Delta)^3}{2}}\right] + b_2$$

$$\Rightarrow\ b_2 = -V_R\frac{(L-\Delta)^3}{6} - EIh \qquad 1$$

$$\therefore\ v(x) = \frac{V_R}{EI}\left[(L-\Delta)\frac{x^2}{2} - \frac{x^3}{6} - \frac{(L-\Delta)^2 x}{2} - \frac{(L-\Delta)^3}{6}\right] - h$$

Disp at $x = \frac{L}{2}+$

$$v\left(x=\frac{L}{2}^+\right) = \frac{V_R}{EI}\left[(L-\Delta)\frac{L^2}{8} - \frac{L^3}{48} - \frac{(L-\Delta)^2 L}{4} - \frac{(L-\Delta)^3}{6}\right] - h$$

Slope at $x = \frac{L}{2}+$

$$\frac{dv}{dx}\left(x=\frac{L}{2}+\right) = \frac{V_R}{EI}\left[(L-\Delta)\frac{L}{2} - \frac{L^2}{8} - \frac{(L-\Delta)^2}{2}\right]$$

$$= \frac{V_R}{EI}\left[\frac{3L^2}{8} - \frac{\Delta L}{2} - \frac{L^2}{2} + L\Delta - \frac{\Delta^2}{2}\right]$$

Since the beam deflection profile is assumed smooth the deflection & the slope at $x = \frac{L}{2}$ must be same from two equations, i.e.

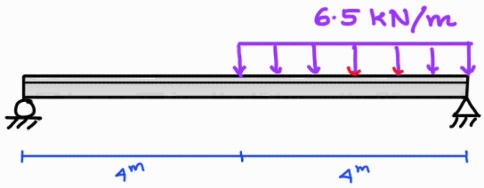
$$\left. v \right|_{x = L/2^-} = \left. v \right|_{x = L/2^+}$$

$$\left. \frac{dv}{dx} \right|_{x = L/2^-} = \left. \frac{dv}{dx} \right|_{x = L/2^+} \qquad (1)$$

From these two equations, one can solve for unknown V_R and $\Delta \leftarrow$ is being asked in question

Q4)

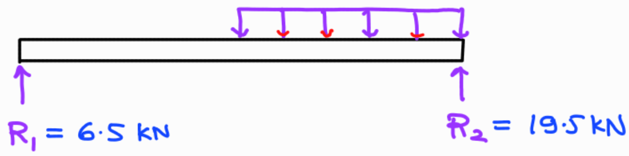
Support reactions



$$\circlearrowleft + \sum M_{left} = 0$$

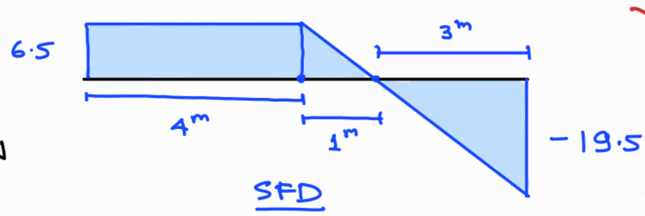
$$\Rightarrow \quad R_2\,(8) - (6.5)(4)\left(\frac{4}{2}+4\right) = 0$$

$$\Rightarrow \quad R_2 = 19.5 \text{ kN}$$



$$+\uparrow \sum F_y = 0 \quad \Rightarrow \quad R_1 + R_2 = (6.5)(4)$$

$$\Rightarrow \quad R_1 = 26 - 19.5 = 6.5 \text{ kN}$$



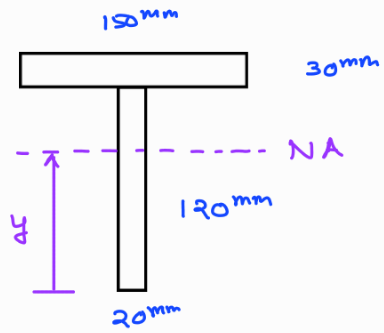
$1\frac{1}{2}$

Maximum shear force will occur at the right support

Section properties

The NA lies at the centroid. The centroid of the c/s has to be determined.

Considering the reference at the bottom-most fiber, the centroid $\bar{y}$ is



1

$$\bar{y} = \frac{(150)(30)\left(\frac{30}{2}+120\right) + (120)(20)\left(\frac{120}{2}\right)}{(150)(30) + (120)(20)}$$

$$= 108.91 \text{ mm}$$

The moment of inertia about the neutral axis:

1

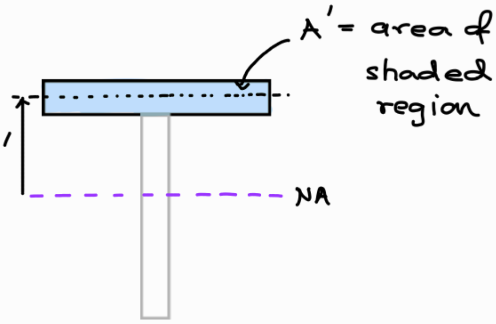
$$I_{zz} = \left[\frac{1}{12}(0.15)(0.03)^3 + (0.15)(0.03)\left(\frac{0.03}{2} + 0.12 - 0.10891\right)^2 + \frac{1}{12}(0.02)(0.12)^3 + (0.12)(0.02)\left(0.10891 - \frac{0.12}{2}\right)^2\right]$$

$$= 3.4 \times 10^{-6} \text{ m}^4 + 8.621 \times 10^{-6} \text{ m}^4$$

$$= 1.2021 \times 10^{-5} \text{ m}^4$$

It is the glue's resistance to longitudinal shear stress at the connection that holds the boards from slipping at the right hand support.

Shear stress at the level of the connection



1.5

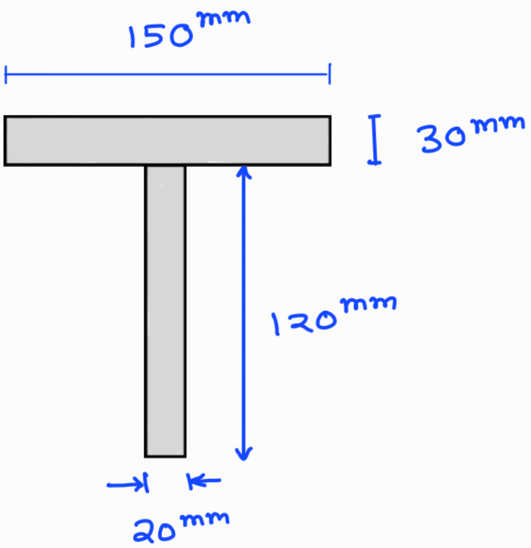
$$Q = \bar{y}' A'$$

$$= \left(0.12 + \frac{0.03}{2} - 0.10891\right)(0.15)(0.03)$$

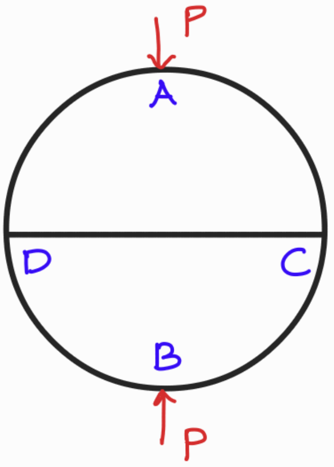
$$= 1.17405 \times 10^{-4} \ m^3$$

1

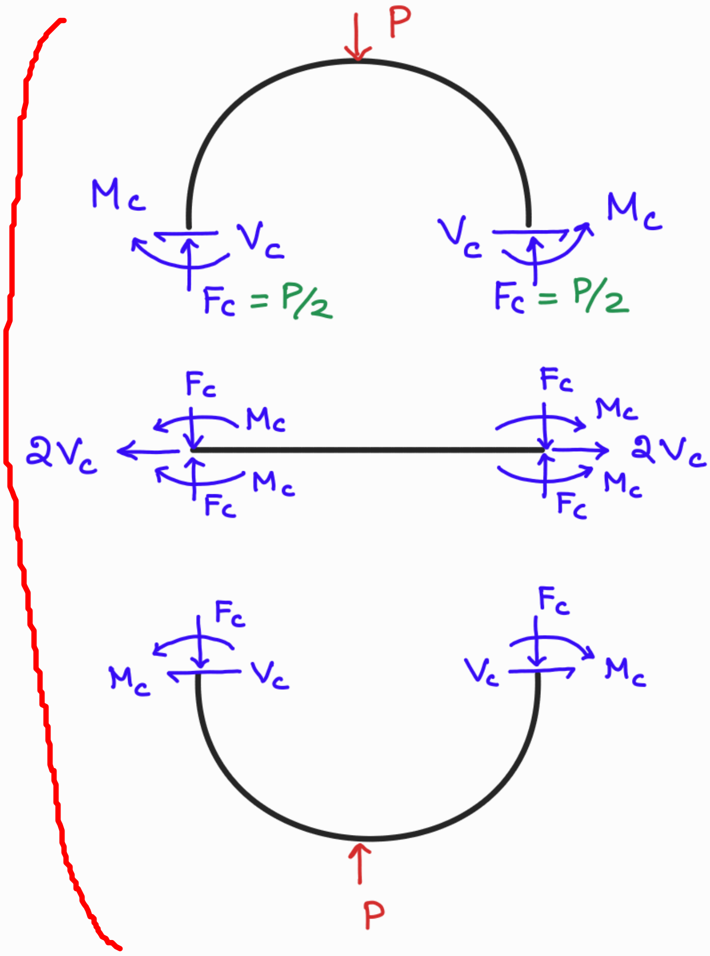
$$\tau_{max} = \frac{V_{max}\, Q}{I_{zz}\, b_{min}} = \frac{(19.5 \times 10^3 \ N)\,(1.17405 \times \cancel{10^{-4}} \ m^3)}{(1.2021 \times \underset{10^{-1}}{\cancel{10^{-5}}} \ m^4)\,(0.02 \ m)} = 9.52 \ MPa$$



Q5>

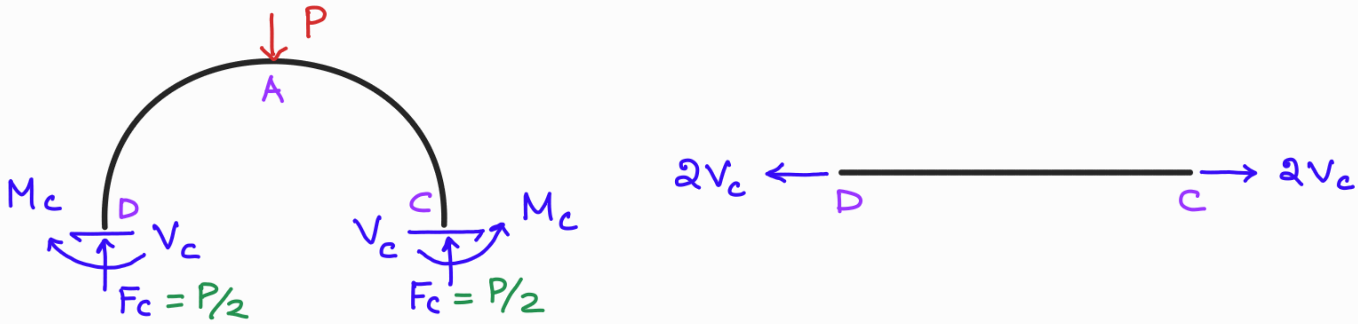


1



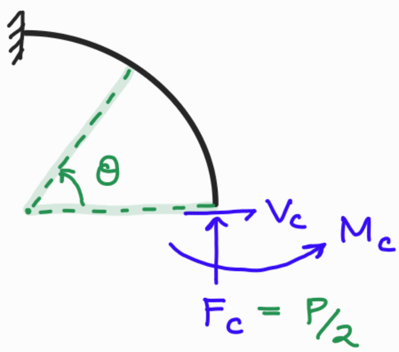
Let AB move closer to each other by δ_{AB} and CD move apart from each other by δ_{CD}

FBD

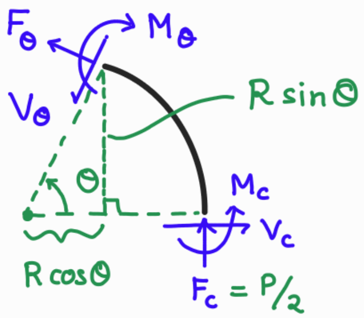


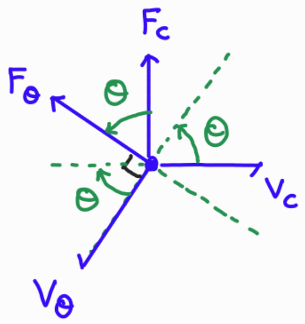
* The straight member is subjected to purely tensile load $2V_c$
* V_c is not zero unlike in the tutorial problem — 1/2
* No rotation at C implies $\dfrac{\partial U_i}{\partial M_c} = 0$

1/2



Balance of forces & moment





1.5

$$\begin{cases} F_\theta = V_c \sin\theta - F_c \cos\theta \\ V_\theta = V_c \cos\theta + F_c \sin\theta \\ M_\theta = M_c + V_c R\sin\theta + F_c R(1-\cos\theta) \end{cases}$$

We will write the internal strain energy as $U_i(M_c, V_c, F_c)$

out of which $F_c = P/2$ (known) and M_c, V_c are unknown!

Using the straight member, we can get V_c. We know the axial elongation of the member CD is $\frac{PL}{AE}$, i.e.

$$\delta_{CD} = \frac{(2V_c)(2R)}{AE} = \frac{4V_cR}{AE}$$

Now using the energy relation that

1

$$\frac{\partial U_i}{\partial V_c} = \frac{\delta_{CD}}{2} = \frac{2V_cR}{AE}$$

you can get the value of V_c

Now for the quarter ring, we can write the internal strain energy due to axial and bending (shear could be neglected)

1

$$U_i = \int_0^{\pi/2} \frac{F_\theta^2 \, R d\theta}{2AE} + \int_0^{\pi/2} \frac{M_\theta^2}{2EI} R d\theta$$

$$= \int_0^{\pi/2} \left[\frac{\left(V_c \sin\theta - F_c \cos\theta\right)^2}{2AE} + \frac{\left(M_c + V_c R\sin\theta + F_c R(1-\cos\theta)\right)^2}{2EI} \right] R d\theta$$

$$\frac{\partial U_i}{\partial M_c} = \int_0^{\pi/2} \frac{\left(M_c + V_c R\sin\theta + F_c R(1-\cos\theta)\right)}{EI} R\, d\theta = 0$$

1

$$\Rightarrow M_c\left(\frac{\pi}{2}\right) - V_c R \cos\theta\Big|_0^{\pi/2} + F_c R\left(\frac{\pi}{2} - \sin\theta\Big|_0^{\pi/2}\right) = 0$$

$$\Rightarrow M_c\left(\pi/2\right) + V_c R + \cancelto{P/2}{F_c} R\left(\frac{\pi}{2} - 1\right) = 0$$

$$\Rightarrow M_c = -\frac{2V_c R}{\pi} - \frac{PR}{2}\left(1 - \frac{2}{\pi}\right) \quad \text{——— (1)}$$

$$\frac{\partial U_i}{\partial V_c} = \int_0^{\pi/2} \left[\left(\frac{V_c \sin\theta - \cancelto{P/2}{F_c} \cos\theta}{AE}\right)\sin\theta + \frac{\left(M_c + V_c R\sin\theta + \cancelto{P/2}{F_c} R(1-\cos\theta)\right)}{EI} R\sin\theta \right] R d\theta = \frac{\delta_{CD}}{2} \quad \text{——— (2)}$$

1

Solving 1 and 2 will give V_c and M_c !

Finally, solving $\left.\dfrac{\partial U_i}{\partial F_c}\right|_{F_c = P/2}$ will give $\dfrac{\delta_{AB}}{2}$

1