Tutorial 5: Mohr's circle

APL 104 - 2022 (Solid Mechanics)

1. The stress tensor at a point is given by the following matrix in Cartesian coordinate system:

$$\left[\underline{\underline{\sigma}}\right] = \begin{bmatrix} -4 & 4 & 0 \\ 4 & -4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

   (a) Draw Mohr's circle corresponding to this state for traction on planes whose normals lie in $(x - y)$ plane. What are the principal stress components and the corresponding principal normals? What is the maximum shear traction and on what plane does it act?

   (b) Using Mohr's circle idea, find out the normal and shear tractions on a plane whose normal lies in $(x - y)$ plane and makes an angle of $7.5°$ from x-axis in clockwise direction.

2. The stress tensor at a point is denoted by the following matrix in Cartesian coordinate system:

$$\left[\underline{\underline{\sigma}}\right] = \begin{bmatrix} -7 & 6\sqrt{3} & 0 \\ 6\sqrt{3} & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

   (a) Draw Mohr's circle corresponding to this state for tractions in $(x - y)$ plane. What are the principal stress components and the direction of principal planes? What is the maximum shear traction and on what plane does it act?

   (b) Using Mohr's circle idea, find out the normal and shear tractions on a plane whose normal lies in $(x - y)$ plane and makes an angle of $15°$ from x-axis in clockwise direction.

   (c) Find out the octahedral normal and shear stress components corresponding to this state of stress.

   (d) Decompose the given stress matrix into hydrostatic and deviatoric part.

3. Suppose the state of stress at a point is as follows in $(x - y - z)$ coordinate system.

$$\left[\underline{\underline{\sigma}}\right] = \begin{bmatrix} -2 & 4\sqrt{3} & 0 \\ 4\sqrt{3} & 6 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

   (a) Find out the center and radius of corresponding Mohr's circle.

(b) Find out (σ, τ) on a plane whose normal makes an angle $15°$ anti-clockwise from x-axis.

(c) What are the values of the principal stress components?

(d) Obtain the orientation of principal stress planes.

APL 104 Tutorial 5 solutions

Q1. The stress tensor at a point is given by the following matrix in Cartesian coordinate system:

$$\left[\underline{\underline{\sigma}}\right] = \begin{bmatrix} -4 & 4 & 0 \\ 4 & -4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

(a) Draw Mohr's circle corresponding to this state for traction on planes whose normals lie in $(x-y)$ plane. What are the principal stress components and the corresponding principal normals? What is the maximum shear traction and on what plane does it act?

(b) Using Mohr's circle idea, find out the normal and shear tractions on a plane whose normal lies in $(x-y)$ plane and makes an angle of 7.5° from x-axis in the clockwise direction.

Solution: The center of the Mohr's circle will be at $(-4,0)$. The x- plane will be at $(-4,4)$, whereas the y-plane will be at $(-4,-4)$. The principal stress components are $\lambda_1 = -8$, $\lambda_2 = 0$, and $\lambda_3 = 3$. The maximum shear is $\tau^{\max} = 4$, and the plane of maximum shear is aligned along e_1-plane.

Q2. The stress tensor at a point is denoted by the following matrix in the Cartesian coordinate system:

$$\left[\underline{\underline{\sigma}}\right] = \begin{bmatrix} -7 & 6\sqrt{3} & 0 \\ 6\sqrt{3} & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

(a) Draw Mohr's circle corresponding to this state for tractions in $(x-y)$ plane. What are the principal stress components and the direction of principal planes? What is the maximum shear traction and on what plane does it act?

(b) Using Mohr's circle idea, find out the normal and shear tractions on a plane whose normal lies in $(x-y)$ plane and make an angle of 15° from x-axis in a clockwise direction.

(c) Find out the octahedral normal and shear stress components corresponding to this state of stress.

(d) Decompose the given stress matrix into the hydrostatic and deviatoric parts.

Solution: The e_3-axis (or the z-axis) corresponds to the principal axis and hence the stress tensor $\underline{\underline{\sigma}}$ can be readily represented by a Mohr's circle in $x-y$ plane.

(a) The Mohr's circle is drawn in Fig. 1.

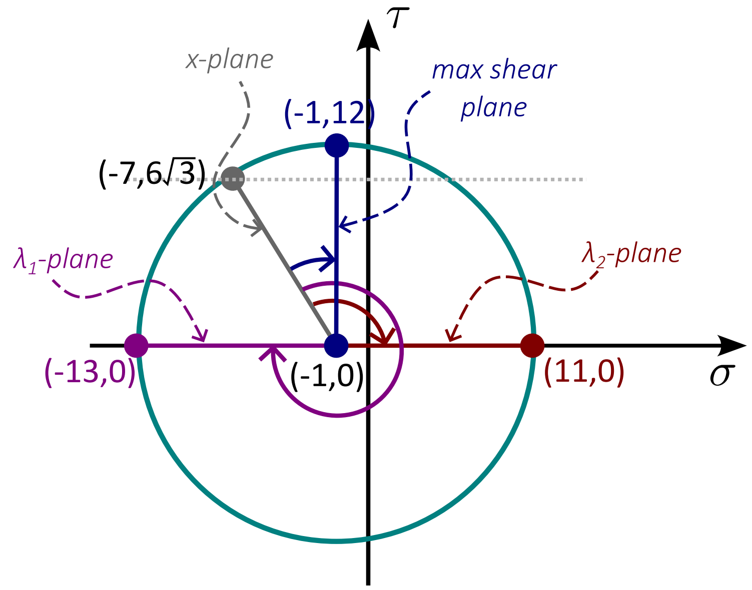

Figure 1: Mohr's circle for Q2(a)

- Center: $\left(\frac{\sigma_{11}+\sigma_{22}}{2}, 0\right) = (-1, 0)$
- Draw point on circle corresponding to $\underset{\sim}{e}_1$-plane with coordinates $(-7, 6\sqrt{3})$
- Radius of Mohr's circle: $R = \sqrt{(-7+1)^2 + (6\sqrt{3})^2} = 12$
- Principal Stresses:

$$\lambda_1 = -1 - 12 = -13 \text{ (extreme left point)}$$
$$\lambda_2 = -1 + 12 = 11 \text{ (extreme right point)}$$
$$\lambda_3 = 3 \text{ (already known)}$$

- Principal plane normals:
  (i) λ_1-plane occurs at $(360° - 60°) = 300°$ clockwise from the $\underset{\sim}{e}_1-$plane in Mohr's circle $\Leftrightarrow$ λ_1-plane occurs at 150° anti-clockwise from the $\underset{\sim}{e}_1-$plane in physical coordinate system $\implies \alpha = 150°$.

$$[\underset{\sim}{n}_1] = \begin{bmatrix} \cos\alpha \\ \sin\alpha \\ 0 \end{bmatrix} = \begin{bmatrix} \cos 150° \\ \sin 150° \\ 0 \end{bmatrix} = \begin{bmatrix} -\cos 30° \\ \sin 30° \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix}$$

  (ii) λ_2-plane occurs at $(180° - 60°) = 120°$ clockwise from the $\underset{\sim}{e}_1-$plane in Mohr's circle $\Leftrightarrow$ λ_2-plane occurs at 60° anti-clockwise from the $\underset{\sim}{e}_1-$plane in physical coordinate system $\implies \alpha = 60°$.

$$[\underset{\sim}{n}_2] = \begin{bmatrix} \cos\alpha \\ \sin\alpha \\ 0 \end{bmatrix} = \begin{bmatrix} \cos 60° \\ \sin 60° \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \\ 0 \end{bmatrix}$$

  (iii) λ_3-plane occurs along $\underset{\sim}{e}_3-$plane, as already mentioned.

- Maximum shear stresses and plane normals

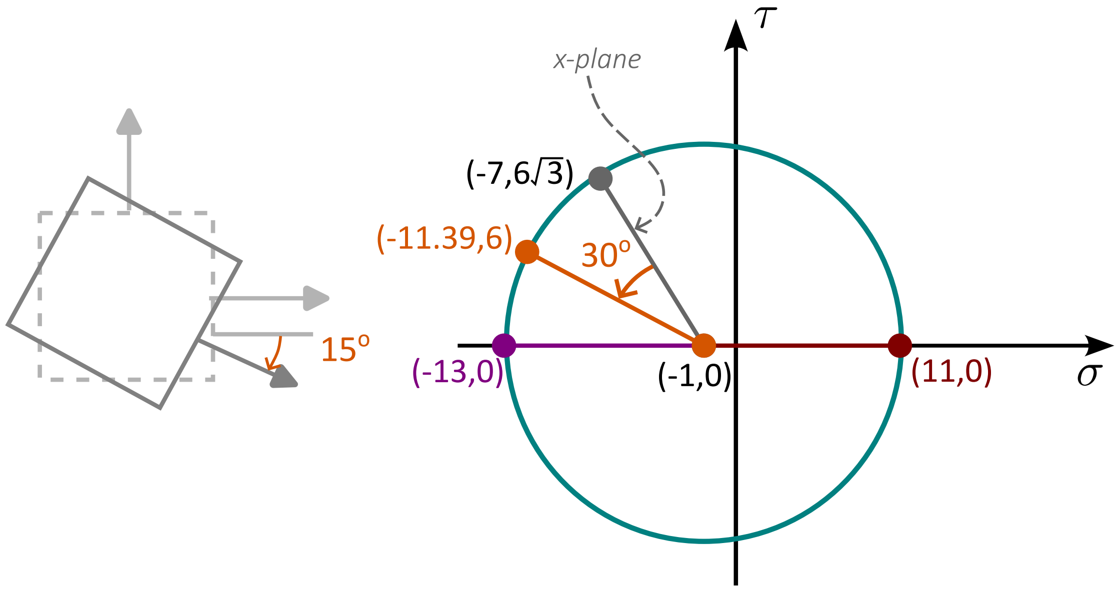

Figure 2: Mohr's circle for Q2(b)

(a) $\tau^{(1)}_{max} = \left|\frac{\lambda_1-\lambda_2}{2}\right| = \left|\frac{-13-11}{2}\right| = 12$ which lies at 30° clockwise in Mohr's circle and hence will be at 15° anti-clockwise from $\underline{e}_1$-plane in physical coordinate system.

$\left[\underline{n}^{(1)}\right] = \pm \begin{bmatrix}\cos 15^\circ \\ \sin 15^\circ \\ 0\end{bmatrix} = \pm \begin{bmatrix}0.9659 \\ 0.2588 \\ 0\end{bmatrix}.$

(b) $\tau^{(2)}_{max} = \left|\frac{\lambda_1-\lambda_3}{2}\right| = \left|\frac{-13-3}{2}\right| = 8.$

To derive $\tau^{(2)}_{max}$ using Mohr's circle, we would need to draw the Mohr's circle corresponding to $\lambda_1 - \lambda_3$ plane. Alternatively, we know that max-shear stress plane occurs at 45° from the two principal planes. The normal of λ_1-plane was obtained earlier. Therefore, this max-shear plane will have the following normal:

$\left[\underline{n}^{(2)}\right] = \frac{1}{\sqrt{2}}\left[\begin{bmatrix}-\frac{\sqrt{3}}{2} \\ \frac{1}{2} \\ 0\end{bmatrix} + \begin{bmatrix}0 \\ 0 \\ 1\end{bmatrix}\right] = \frac{1}{\sqrt{2}}\begin{bmatrix}-\frac{\sqrt{3}}{2} \\ \frac{1}{2} \\ 1\end{bmatrix}.$

There will be three more planes on which this shear stress will be realized.

(c) $\tau^{(3)}_{max} = \left|\frac{\lambda_2-\lambda_3}{2}\right| = \left|\frac{11-3}{2}\right| = 4.$

The corresponding plane normal can be derived as we did in part(b) above.

(b) 15° clockwise from $\underline{e}_1-$plane physically $\Leftrightarrow$ 30° anticlockwise from $\underline{e}_1-$plane in Mohr's circle. Hence

$$\sigma = -1 - R\cos 30^\circ = -1 - 12\frac{\sqrt{3}}{2} = -11.39$$
$$\tau = R\sin 30^\circ = 12.\frac{1}{2} = 6$$

(c) Octahedral normal stress:

$$\sigma_{oct} = \frac{1}{3}I_1 = \frac{1}{3}\frac{(\sigma_{11}+\sigma_{22}+\sigma_{33}}{3}) = \frac{1}{3}(-7+5+3) = \frac{1}{3}.$$

Octahedral shear stress:

$$
\begin{aligned}
\tau_{oct} &= \frac{1}{3}\sqrt{2{I_1}^2 - 6I_2} \\
I_2 &= \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} - {\tau_{12}}^2 - {\tau_{23}}^2 - {\tau_{31}}^2 \\
&= -149 \\
\tau_{oct} &= \frac{1}{3}\sqrt{2\left(\frac{1}{3}\right)^2 - 6(-149)} \\
&= 9.977.
\end{aligned}
$$

(d) Decomposition of stress tensor:

$$
\begin{aligned}
\left[\underline{\underline{\sigma_h}}\right] &= \frac{1}{3}\left[\underline{\underline{I}}\right] = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \\
\left[\underline{\underline{\sigma_d}}\right] &= \left[\underline{\underline{\sigma}}\right] - \left[\underline{\underline{\sigma_h}}\right] \\
&= \begin{bmatrix} -7 - \frac{1}{3} & 6\sqrt{3} & 0 \\ 6\sqrt{3} & 5 - \frac{1}{3} & 0 \\ 0 & 0 & 3 - \frac{1}{3} \end{bmatrix} \\
&= \begin{bmatrix} -\frac{22}{3} & 6\sqrt{3} & 0 \\ 6\sqrt{3} & \frac{14}{3} & 0 \\ 0 & 0 & \frac{8}{3} \end{bmatrix}.
\end{aligned}
$$

Q3. Suppose the state of stress at a point is as follows in $(x-y-z)$ coordinate system.

$$
\left[\underline{\underline{\sigma}}\right] = \begin{bmatrix} -2 & 4\sqrt{3} & 0 \\ 4\sqrt{3} & 6 & 0 \\ 0 & 0 & 4 \end{bmatrix}
$$

(a) Find out the center and radius of corresponding Mohr's circle.

(b) Find out (σ, τ) on a plane whose normal makes an angle 15° anti-clockwise from x-axis.

(c) What are the values of the principal stress components?

(d) Obtain the orientation of principal stress planes.

Solution:

(a)
- Center: $\left(\frac{\sigma_{11}+\sigma_{22}}{2}, 0\right) = \left(\frac{-2+6}{2}, 0\right) = (2, 0)$
- Draw point on circle corresponding to the $\underline{e}_1$-plane, with coordinates $(-2, 4\sqrt{3})$

- Radius of Mohr's circle, $R = \sqrt{(-2-2)^2 + (4\sqrt{3})^2} = 8$

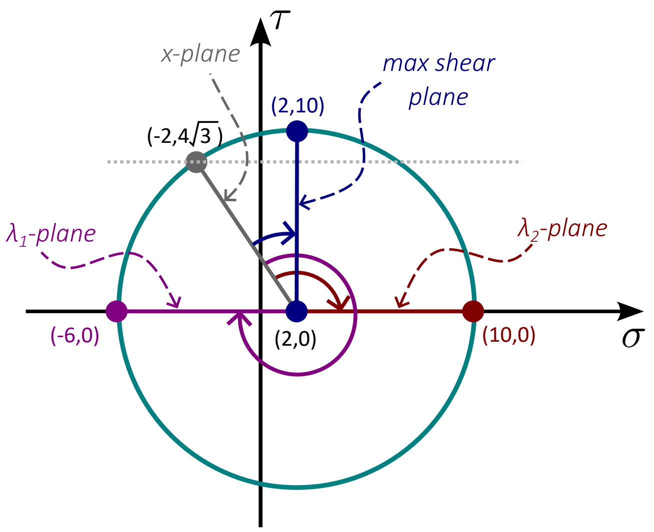

Figure 3: Mohr's circle for Q3(a)

(b) 15° anti-clockwise from $\underline{e}_1$ physically $\Leftrightarrow$ 30° clockwise from $\underline{e}_1-$plane in Mohr's circle, which coincides with the plane of maximum shear, as can be seen from Fig. 4. The stresses on this plane is $(\sigma, \tau) = (2, 8)$.

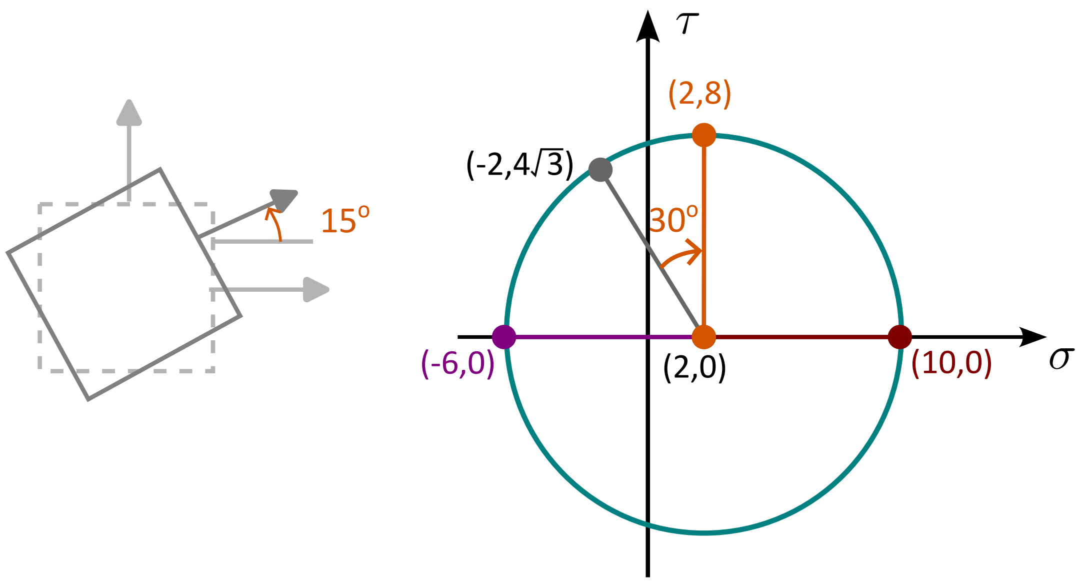

Figure 4: Mohr's circle for Q3(b)

(c) Principal stresses:

$$\lambda_1 = 2 + 8 = 10 \text{ (Extreme right point)}$$
$$\lambda_2 = 2 - 8 = -6 \text{ (Extreme left point)}$$
$$\lambda_3 = 4 \text{ (as given)}$$

(d) Orientation of principal planes: can be obtained as in previous problem.