Deformation and strain

Three relations govern behavior of a solid deformable body:

a) Force equilibrium relations (Stress equilibrium PDEs)

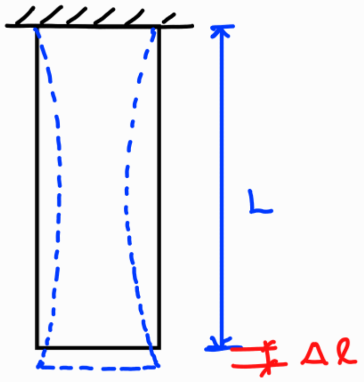
6 independent unknowns and 3 equations

b) Geometric compatibility ⟹ (Strain-compatibility PDEs)

[deformation should occur in such a fashion that there is no overlap or void created in the deformed body]

c) Force-deformation relations ⟹ Stress-strain behavior

Consider an example of a bar hanging under the action of self-weight

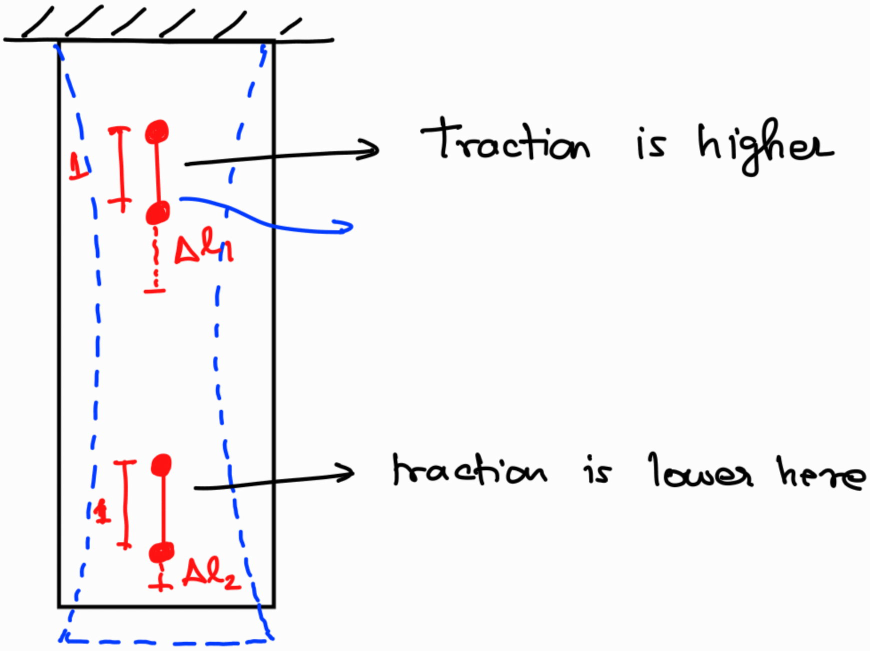


How do you define say longitudinal strain ?

$$\text{Avg longitudinal strain} = \frac{\Delta \ell}{L}$$

$$= \frac{\Delta \ell}{(L + \Delta \ell)}$$

In general, "strain" in a body varies from point to point.



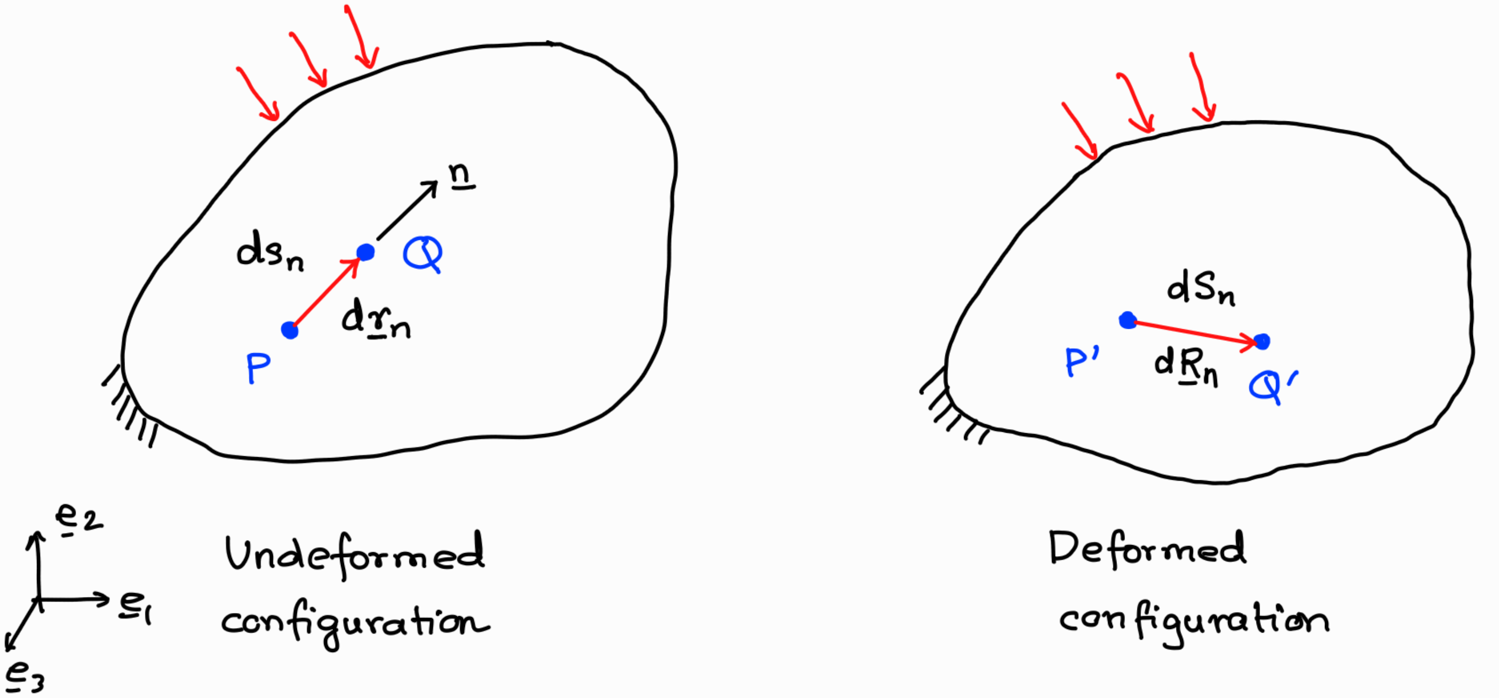
Definition of strain components

Like stresses, we have two types of strains → normal strain / → shear strain

Let's see how are they defined.

Normal Strain

When a body deforms, line elements inside the body either elongate or shrink. NORMAL STRAIN is used to measure the change in length of a small line element at a point inside the body



Undeformed configuration

Deformed configuration

$$E_{nt} = \frac{1}{2} \frac{(dS_n)^2 - (ds_n)^2}{(ds_n)^2}$$

Normal strain

$$(dS_n)^2 = d\underline{R}_n \cdot d\underline{R}_n$$

$$(ds_n)^2 = d\underline{r}_n \cdot d\underline{r}_n$$

$$= \frac{1}{2} \frac{d\underline{R}_n \cdot d\underline{R}_n - d\underline{r}_n \cdot d\underline{r}_n}{(ds_n)^2}$$

$$= \frac{1}{2}\left(\frac{d\underline{R}_n}{ds_n} \cdot \frac{d\underline{R}_n}{ds_n} - \cancelto{\underline{n}}{\frac{d\underline{r}_n}{ds_n}} \cdot \cancelto{\underline{n}}{\frac{d\underline{r}_n}{ds_n}} \right)$$

$$= \frac{1}{2}\left(\underbrace{\frac{d\underline{R}_n}{ds_n} \cdot \frac{d\underline{R}_n}{ds_n}}_{} - 1 \right)$$

$$\boxed{E_{nn} = \frac{1}{2}\left(\left\| \frac{d\underline{R}_n}{ds_n} \right\|^2 - 1 \right)}$$

$$\frac{d\underline{R}_n}{ds_n} \cdot \frac{d\underline{R}_n}{ds_n}$$

$$\left\| \frac{d\underline{R}_n}{ds_n} \right\| = \sqrt{2E_{nn}+1}$$

Linearized normal strain

$$\epsilon_{nn} = \frac{1}{2}\,\frac{dS_n^2 - ds_n^2}{ds_n^2} = \frac{1}{2}\,\frac{(dS_n - ds_n)(\cancelto{ds_n}{dS_n} + ds_n)}{ds_n^2}$$

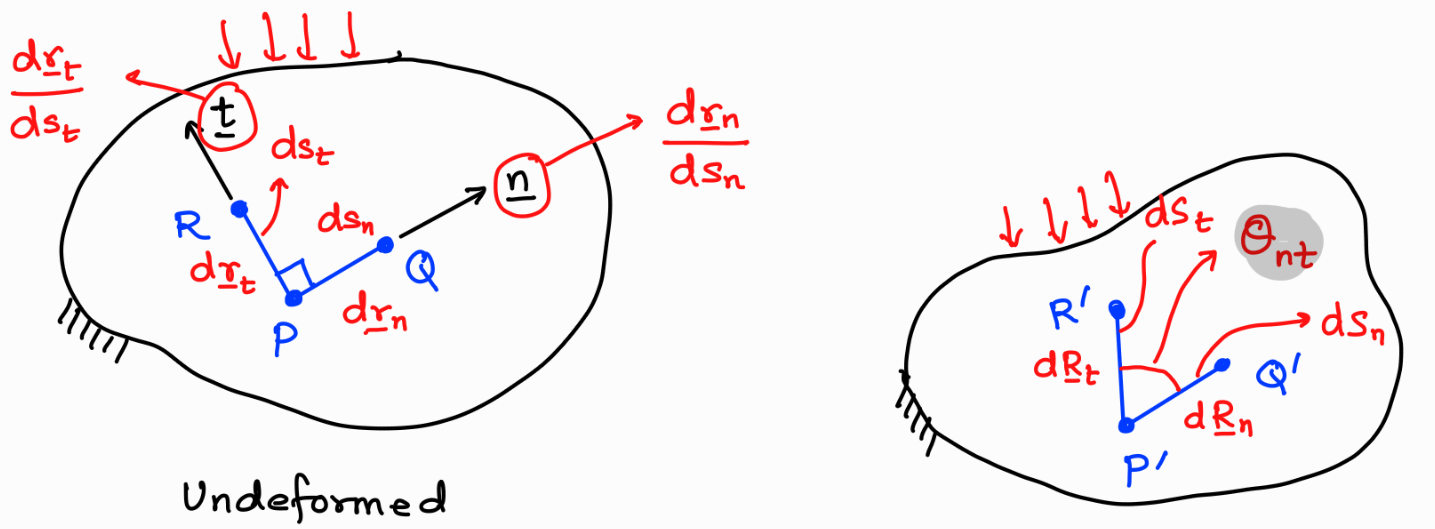
$$= \frac{1}{\cancel{2}}\,\frac{(dS_n - ds_n)(\cancel{2}\cancel{ds_n})}{ds_n^{\cancel{2}}}$$

(Infinitesimal normal)

Linearized normal strain at point P

$$\boxed{\epsilon_{nn} = \lim_{ds_n \to 0} \frac{dS_n - ds_n}{ds_n}}$$

Shear strain



Undeformed config

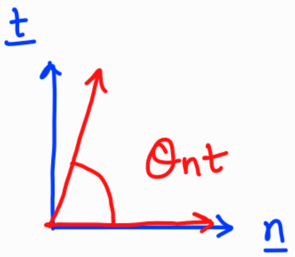
Deformed config

$$E_{nt} = \frac{1}{2} \boxed{\underline{\frac{dR_n}{ds_n} \cdot \frac{dR_t}{ds_t}}}$$

$$\cos \theta_{nt} = \frac{\dfrac{dR_n}{ds_n} \cdot \dfrac{dR_t}{ds_t}}{\left\| \dfrac{dR_n}{ds_n} \right\| \left\| \dfrac{dR_t}{ds_t} \right\|}$$

$$\left\| \frac{dR_n}{ds_n} \right\|$$

$$\cos \theta_{nt} = \frac{\dfrac{dR_n}{ds_n} \cdot \dfrac{dR_t}{ds_t}}{\sqrt{2E_{nn}+1}\ \sqrt{2E_{tt}+1}} = \frac{2E_{nt}}{\sqrt{2E_{nn}+1}\ \sqrt{2E_{tt}+1}}$$



$$\cos \theta_{nt} = \underline{\sin\left(\frac{\pi}{2} - \theta_{nt}\right)} \approx \frac{\pi}{2} - \theta_{nt}$$

when $\theta_{nt} \sim \frac{\pi}{2}$

$$\theta_{nt} \approx \frac{\pi}{2} \Rightarrow \frac{\pi}{2} - \theta_{nt} \approx 0$$

$$\frac{\pi}{2} - \theta_{nt} = \frac{2E_{nt}}{\sqrt{2E_{nn}+1}\ \sqrt{2E_{tt}+1}}$$

$E_{nn} << 1$

$E_{tt} << 1$

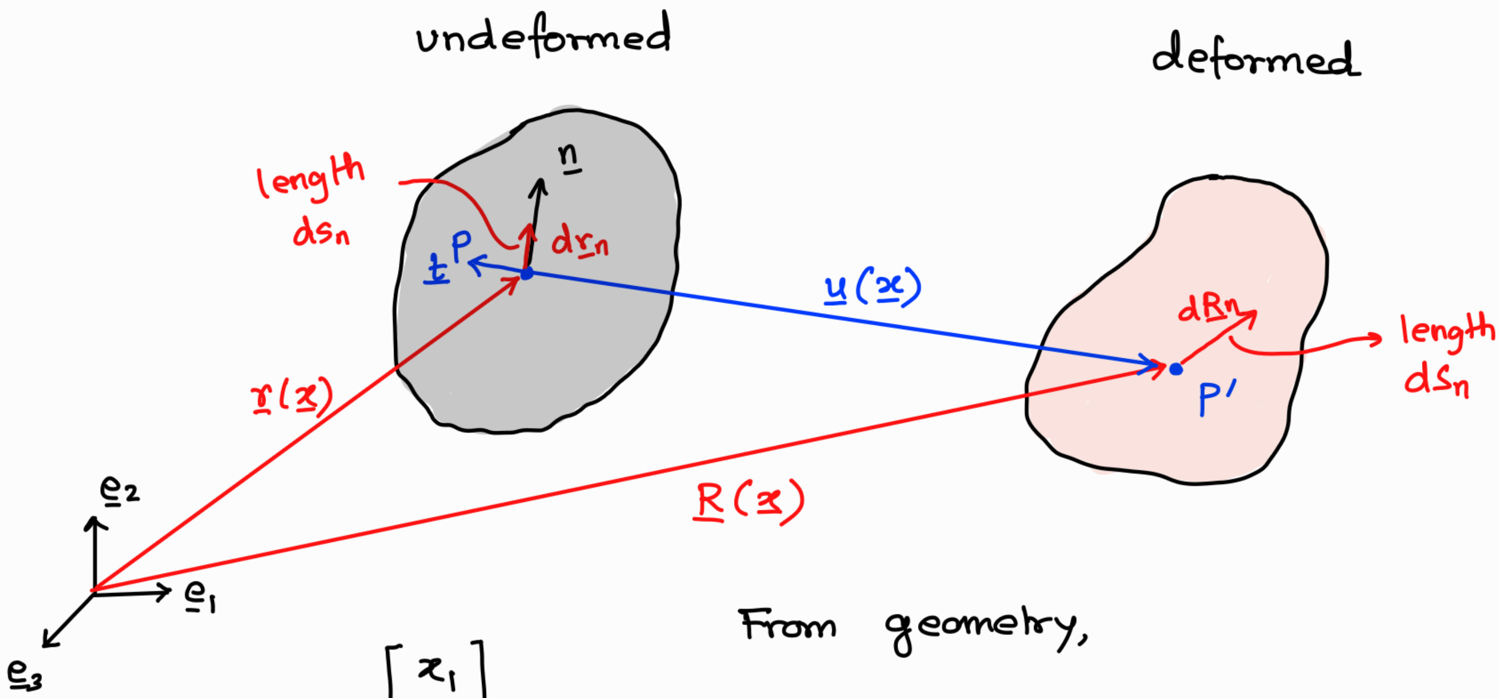
$$E_{nt} = \frac{1}{2}\left(\frac{\pi}{2} - \theta_{nt}\right) \sqrt{2E_{nn}+1}\ \sqrt{2E_{tt}+1}$$

The strains are very small, $E_{nn} \approx 0$, $E_{tt} \approx 0$

Linearized shear strain $\quad \epsilon_{nt} = \frac{1}{2}\left(\frac{\pi}{2} - \theta_{nt}\right)$

Engineering shear strain $\quad \gamma_{nt} = 2\epsilon_{nt} = \left(\frac{\pi}{2} - \theta_{nt}\right)$

Strain- displacement relations



$$[\underline{x}]_{\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

From geometry,

$$\underline{R}(\underline{x}) = \underline{r}(\underline{x}) + \underline{u}(\underline{x})$$

$$\Rightarrow \quad \frac{d\underline{R}_n(\underline{x})}{ds_n} = \frac{d\underline{r}_n(\underline{x})}{ds_n} + \frac{d\underline{u}(\underline{x})}{ds_n}$$

$$\Rightarrow \quad \frac{d\underline{R}_n}{ds_n} = \overbrace{\frac{d\underline{r}_n}{ds_n}}^{\underline{n}} + \frac{d\underline{u}}{ds_n}$$

$$\Rightarrow \quad \frac{d\underline{R}_n}{ds_n} = \underline{n} + \frac{d\underline{u}}{ds_n}$$

$$\frac{d\underline{R}_t}{ds_t} = \underline{t} + \frac{d\underline{u}}{ds_t}$$

$$E_{nn} = \frac{1}{2}\left(\frac{d\underline{R}_n}{ds_n} \cdot \frac{d\underline{R}_n}{ds_n} - 1\right)$$

$$= \frac{1}{2}\left(\left[\underline{n} + \frac{d\underline{u}}{ds_n}\right] \cdot \left[\underline{n} + \frac{d\underline{u}}{ds_n}\right] - 1\right)$$

Using this result in the definition of normal strain

$$E_{nn} = \frac{1}{2}\left(\frac{d\underline{R}_n}{ds_n}\cdot\frac{d\underline{R}_n}{ds_n} - 1\right)$$

$$= \frac{1}{2}\left[\left(\underline{n} + \frac{d\underline{u}}{ds_n}\right)\cdot\left(\underline{n} + \frac{d\underline{u}}{ds_n}\right) - 1\right]$$

$$= \frac{1}{2}\left[\overset{\cancel{1}}{\underline{\underline{n}\cdot\underline{n}}} + 2\,\underline{n}\cdot\frac{d\underline{u}}{ds_n} + \frac{d\underline{u}}{ds_n}\cdot\frac{d\underline{u}}{ds_n} - \cancel{1}\right]$$

$$E_{nn} = \underline{n}\cdot\frac{d\underline{u}}{ds_n} + \frac{1}{2}\frac{d\underline{u}}{ds_n}\cdot\frac{d\underline{u}}{ds_n}$$

If the normal strain E_{nn} is small enough, we can neglect product of displacement gradients:

$$\epsilon_{nn} = \underline{n}\cdot\frac{d\underline{u}}{ds_n}$$

Linearized normal strain

Similarly, if we let $\underline{r}(\underline{x})$ change by a small amount $d\underline{r}_t$ in the unit normal direction $\underline{t}$, we get:

$$\frac{d\underline{R}_t}{ds_t} = \frac{d\underline{r}_t}{ds_t} + \frac{d\underline{u}}{ds_t} = \underline{t} + \frac{d\underline{u}}{ds_t}$$

and using the results in the definition of shear strain

$$E_{nt} = \frac{1}{2}\left(\frac{d\underline{R}_n}{ds_n} \cdot \frac{d\underline{R}_t}{ds_t}\right)$$

$$= \frac{1}{2}\left[\left(\underline{n} + \frac{d\underline{u}}{ds_n}\right) \cdot \left(\underline{t} + \frac{d\underline{u}}{ds_t}\right)\right]$$

$$= \frac{1}{2}\left[\underline{t} \cdot \frac{d\underline{u}}{ds_n} + \underline{n} \cdot \frac{d\underline{u}}{ds_t} + \frac{d\underline{u}}{ds_n} \cdot \frac{d\underline{u}}{ds_t} + \cancelto{0}{\underline{n} \cdot \underline{t}}\right]$$

small

$$= \frac{1}{2}\left[\underline{t} \cdot \frac{d\underline{u}}{ds_n} + \underline{n} \cdot \frac{d\underline{u}}{ds_t} + \frac{d\underline{u}}{ds_n} \cdot \frac{d\underline{u}}{ds_t}\right]$$

If the shear strain E_{nt} is small enough, then the prod. of displacement gradients can be neglected:

$$\epsilon_{nt} = \frac{1}{2}\left[\underline{t} \cdot \frac{d\underline{u}}{ds_n} + \underline{n} \cdot \frac{d\underline{u}}{ds_t}\right] \quad \text{(Shear strain vs disp)}$$

Linearized shear strain

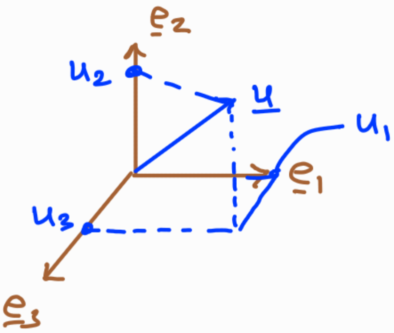
$$\boxed{\epsilon_{nn} = \underline{n} \cdot \frac{d\underline{u}}{ds_n}} \quad \text{(Normal strain vs disp)}$$

$$\epsilon_{11} = \underline{e}_1 \cdot \frac{d\underline{u}}{dx_1} = \underline{e}_1 \cdot \left(\frac{\partial u_1}{\partial x_1}\underline{e}_1 + \frac{\partial u_2}{\partial x_1}\underline{e}_2 + \frac{\partial u_3}{\partial x_1}\underline{e}_3\right)$$

$$= \frac{\partial u_1}{\partial x_1}$$

$$\epsilon_{12} = \frac{1}{2}\left[\underline{e}_2 \cdot \frac{d\underline{u}}{dx_1} + \underline{e}_1 \cdot \frac{d\underline{u}}{dx_2}\right] = \frac{1}{2}\left[\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right]$$

Now if we write the displacement vector $\underline{u}$ using reference coordinate system $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$



$$[\underline{u}]_{\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$\underline{u} = u_1 \underline{e}_1 + u_2 \underline{e}_2 + u_3 \underline{e}_3 = \sum_{i=1}^{3} u_i \underline{e}_i$$

The gradients of $\underline{u}$ w.r.t x_j would be

$$\underline{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \qquad \frac{\partial \underline{u}}{\partial x_j} = \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_j} \underline{e}_i \qquad \text{for } j = 1, 2, 3$$

$$= \partial u_i$$

So the gradient of displacement vector can be written in a matrix form:

$$\underline{\nabla} \underline{u} = \frac{\partial \underline{u}}{\partial \underline{x}} = \frac{\partial \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}}{\partial \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}} = \begin{array}{c c} & \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{array}{c} u_1 \\ u_2 \\ u_3 \end{array} & \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix} \end{array}$$

$$\underline{\nabla} = \left[\frac{\partial}{\partial x_1}(\cdot), \ \frac{\partial}{\partial x_2}(\cdot), \ \frac{\partial}{\partial x_3}(\cdot) \right]$$

$\underline{n} = \underline{e}_1 \qquad \epsilon_{11} = e_1 \cdot \frac{\partial \underline{u}}{\partial x_1} = \frac{\partial u_1}{\partial x_1}$

$$\epsilon_{22} = \frac{\partial u_2}{\partial x_2} \qquad \epsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$$

$$\epsilon_{33} = \frac{\partial u_3}{\partial x_3} \qquad \epsilon_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right)$$

$$\epsilon_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right)$$

State of strain at a point

State of strain comprises of nine strain components:

3 normal strain, and 6 shear strain

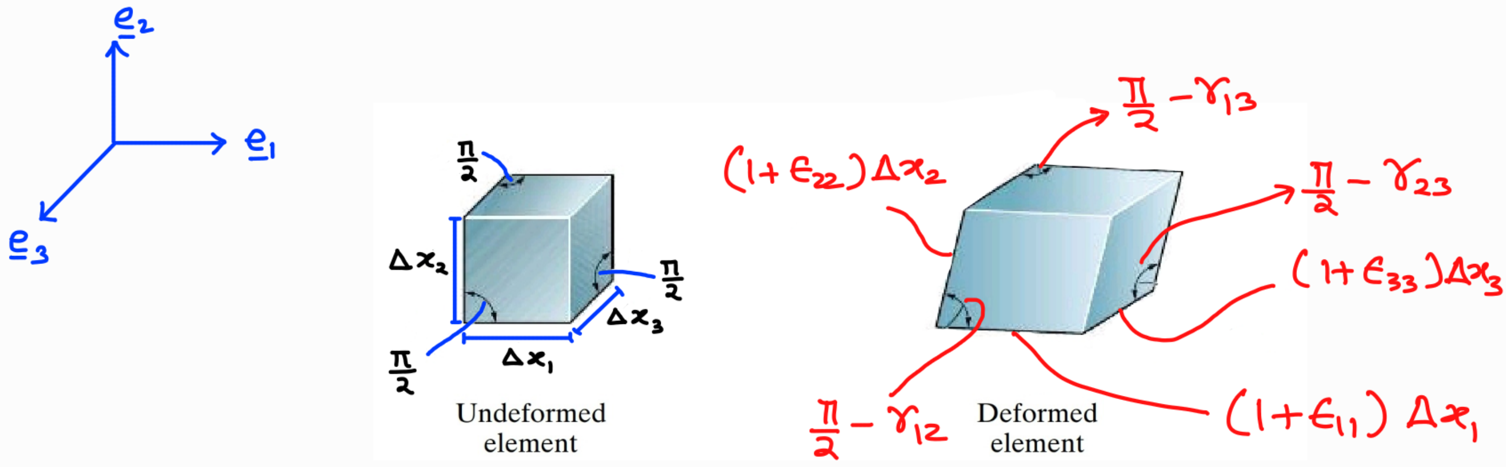
State of strain at a pt $\longrightarrow$ 2nd order strain tensor

$$\underline{\underline{\epsilon}}$$

Strain matrix (dependent upon the coordinate system)

$$[\underline{\underline{\epsilon}}]_{\begin{pmatrix}\underline{e}_1\\ \underline{e}_2\\ \underline{e}_3\end{pmatrix}} = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{12} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{13} & \epsilon_{23} & \epsilon_{33} \end{bmatrix}$$

If the state of strain at a point is known, one can describe the deformation of a small cuboidal element at the point is completely defined by the strain tensor $\underline{\underline{\epsilon}}$



$$\gamma_{12} = \frac{\pi}{2} - \theta_{12}$$

Local average rotation tensor

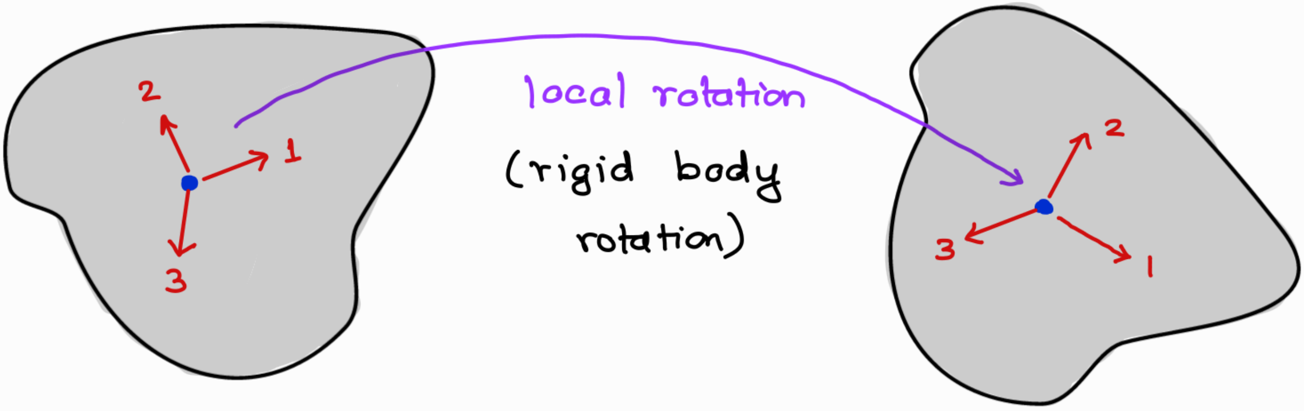
$$\underline{\nabla}\,\underline{u} = \frac{1}{2}\left(\underline{\nabla}\,\underline{u} + \underline{\nabla}\,\underline{u}^T\right) + \frac{1}{2}\left(\underline{\nabla}\,\underline{u} - \underline{\nabla}\,\underline{u}^T\right)$$

$$= \underline{\underline{\epsilon}} + \underline{\underline{W}}$$

small strain tensor (induced at a point)

local average rotation tensor (associated with rigid-body rotation of line elements at that point)

If the strain at a point is zero, then there will be no strain of any kind (normal/shear strain) at that point. However, the line elements can undergo rigid-body rotation.

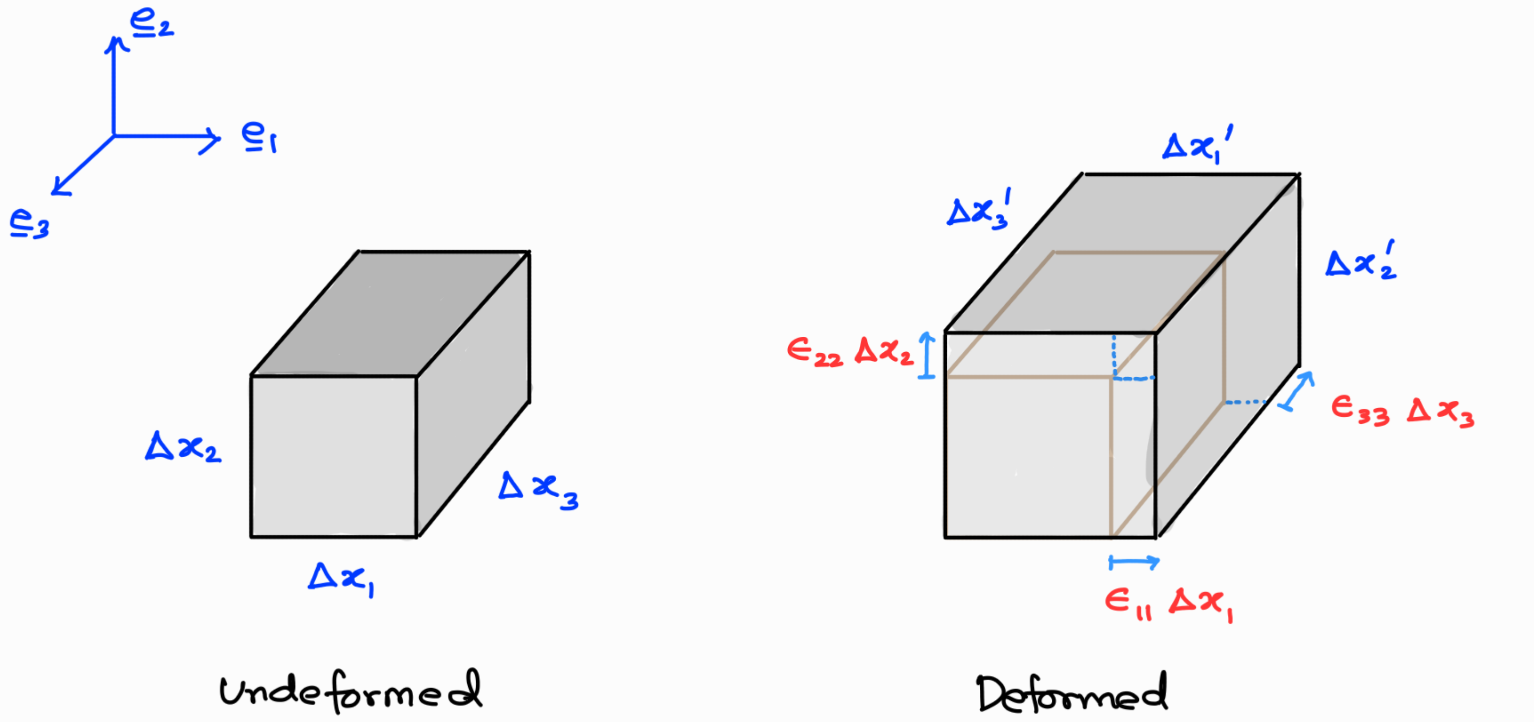


The local average rotation tensor is responsible for rigid-body rotation of line elements.

Local Volumetric Strain (or Dilatational Strain)

As a body deforms, the volume of every small region (called local volume element) of the body also changes.

Volumetric strain : change in volume per unit volume at an infinitisimally small volume element at a point



$$\Delta x_1' = (1+\epsilon_{11})\Delta x_1$$

$$\Delta x_2' = (1+\epsilon_{22})\Delta x_2$$

$$\Delta x_3' = (1+\epsilon_{33})\Delta x_3$$

Volume of original local volume element, $V = \Delta x_1 \Delta x_2 \Delta x_3$

" " element after deformation, $v = \Delta x_1' \Delta x_2' \Delta x_3'$

$$\text{Volumetric strain} = \frac{v - V}{V}$$

$$= \frac{(1+\epsilon_{11})\Delta x_1 \; (1+\epsilon_{22})\Delta x_2 \; (1+\epsilon_{33})\Delta x_3 - \Delta x_1 \Delta x_2 \Delta x_3}{\Delta x_1 \Delta x_2 \Delta x_3}$$

$$= (1+\epsilon_{11})(1+\epsilon_{22})(1+\epsilon_{33}) - 1$$

$$= \cancel{1} + \epsilon_{11} + \epsilon_{22} + \epsilon_{33} - \cancel{1} = \epsilon_{11} + \epsilon_{22} + \epsilon_{33}$$

Strain compatibility equations

The displacement of a point in a deformable body is a vector $\underline{u}$ with three components

$$[\underline{u}]_{(e_1, e_2, e_3)} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

The state of strain at a point is specified by SIX independent strain components

$$\epsilon_{11},\ \epsilon_{22},\ \epsilon_{33},\ \epsilon_{23},\ \epsilon_{13},\ \epsilon_{12}$$

The strain-displacement relations maps six strains from three known displacements, through the derivatives of displacements.

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \underset{\text{How ?}}{\overset{\text{Differentiation}}{\rightleftarrows}} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \epsilon_{23} \\ \epsilon_{13} \\ \epsilon_{12} \end{bmatrix}$$

e.g. $\epsilon_{11} = \dfrac{\partial u_1}{\partial x_1}$

$\epsilon_{13} = \dfrac{1}{2}\left(\dfrac{\partial u_3}{\partial x_1} + \dfrac{\partial u_1}{\partial x_3}\right)$

How do you determine uniquely three displacement functions u_1, u_2, u_3, from six strain components ?

The answer is not straightforward!

Infact, we may not obtain a compatible displacement function by integrating any six arbitrary strain components

Take an example:

$$\epsilon_{11} = 1 , \qquad \epsilon_{22} = x_1^2 , \qquad \epsilon_{12} = x_1^3 x_2$$

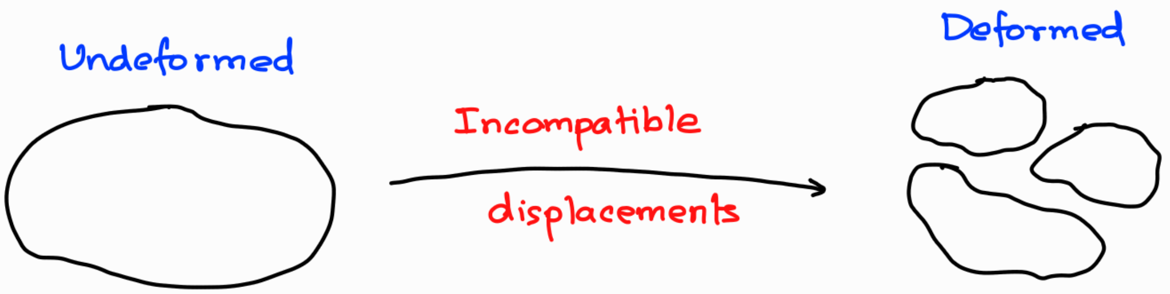
$$\epsilon_{11} = \frac{\partial u_1}{\partial x_1} = 1 \Rightarrow u_1(x_1, x_2, x_3) = x_1 + c_1(x_2, x_3)$$

$$\epsilon_{22} = \frac{\partial u_2}{\partial x_2} = x_1^2 \Rightarrow u_2(x_1, x_2, x_3) = x_1^2 x_2 + c_2(x_1, x_3)$$

The resulting functions u_1 and u_2 obtained from integration of ϵ_{11} & ϵ_{22} maynot satisfy ϵ_{12}

$$\epsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = \frac{\partial c_1}{\partial x_2} + 2x_1 x_2 + \frac{\partial c_2}{\partial x_1}$$

Since 6 strain components are derived from 3 disps. the six strain components can NOT be arbitrary, else six strain components could lead to INCOMPATIBLE displacements.



Thus, there has to be certain constraints on the six strain functions so as to result in a compatible deformation; these constraints are called STRAIN COMPATIBILITY conditions.

There are SIX strain-compatibility conditions (much like the stress-equilibrium relations)

Set 1

$$\frac{\partial^2 \epsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \epsilon_{22}}{\partial x_1^2} = 2\,\frac{\partial^2 \epsilon_{12}}{\partial x_1\,\partial x_2}$$

$$\frac{\partial^2 \epsilon_{22}}{\partial x_3^2} + \frac{\partial^2 \epsilon_{33}}{\partial x_2^2} = 2\,\frac{\partial^2 \epsilon_{23}}{\partial x_2\,\partial x_3}$$

$$\frac{\partial^2 \epsilon_{11}}{\partial x_3^2} + \frac{\partial^2 \epsilon_{33}}{\partial x^2} = 2\,\frac{\partial^2 \epsilon_{13}}{\partial x_1\,\partial x_3}$$

$$\frac{\partial^2}{\partial x_2^2}\left(\frac{\partial u_1}{\partial x_1}\right) + \frac{\partial^2}{\partial x_1^2}\left(\frac{\partial u_2}{\partial x_2}\right) = \frac{\partial^2}{\partial x_1 \partial x_2}\left[\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right]$$

$$2\,\frac{\partial^2 \epsilon_{12}}{\partial x_1\,\partial x_2} = \cancel{2}\,\frac{\partial^2}{\partial x_1 \partial x_2}\left(\frac{1}{\cancel{2}}\left[\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right]\right) = \frac{\partial^2}{\partial x_1 \partial x_2}\left[\ \right]$$

Set 2

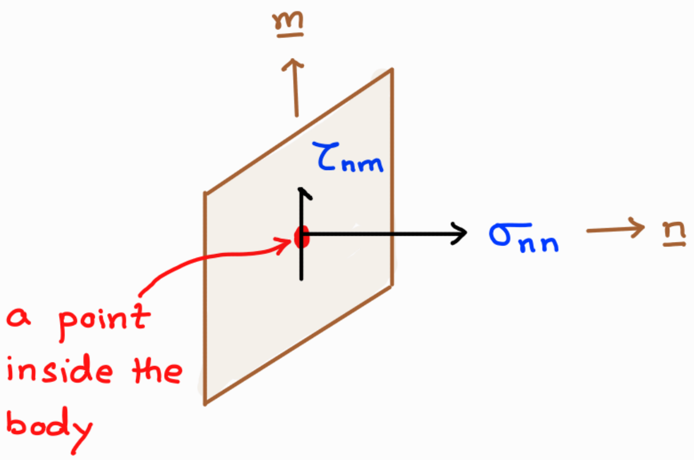
$$\frac{\partial}{\partial x_3}\left(\frac{\partial \epsilon_{23}}{\partial x_1} + \frac{\partial \epsilon_{31}}{\partial x_2} - \frac{\partial \epsilon_{12}}{\partial x_3}\right) = \frac{\partial^2 \epsilon_{33}}{\partial x_1\,\partial x_2}$$

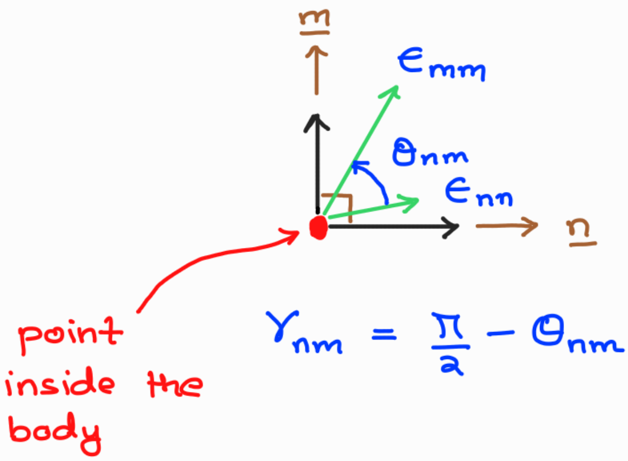
$$\frac{\partial}{\partial x_1}\left(\frac{\partial \epsilon_{12}}{\partial x_3} + \frac{\partial \epsilon_{13}}{\partial x_2} - \frac{\partial \epsilon_{23}}{\partial x_1}\right) = \frac{\partial^2 \epsilon_{11}}{\partial x_2\,\partial x_3}$$

$$\frac{\partial}{\partial x_2}\left(\frac{\partial \epsilon_{12}}{\partial x_3} + \frac{\partial \epsilon_{23}}{\partial x_1} - \frac{\partial \epsilon_{13}}{\partial x_2}\right) = \frac{\partial^2 \epsilon_{33}}{\partial x_1\,\partial x_2}$$

Similarity between stress and strain tensors

Stress	Strain
$\sigma_{nn} = (\underline{\underline{\sigma}}\,\underline{n}) \cdot \underline{n}$	$\epsilon_{nn} = (\underline{\underline{\epsilon}}\,\underline{n}) \cdot \underline{n}$
$\tau_{nm} = (\underline{\underline{\sigma}}\,\underline{n}) \cdot \underline{m}$	$\epsilon_{mn} = (\underline{\underline{\epsilon}}\,\underline{n}) \cdot \underline{m}$
$\underline{n}$: plane normal $\underline{m}$: force intensity direction	$\underline{n}$: direction of line element $\underline{m}$: direction of line element $\perp$ to $\underline{n}$





The previous results obtained for stress transformations (i.e. stress components on an arbitrary inclined plane), principal stresses, Mohr's circle, etc., remain similar for strain as well.

Strain components associated with arbitrary sets of axes

As was the case with plane stress, where we found stress components on an arbitrary face inclined at an angle θ from the $\underline{e}_1$-plane and obtained

$$\sigma_{nn} = (\underline{\underline{\sigma}}\,\underline{n})\cdot\underline{n} = \sigma_{11}\, n_1^2 + \sigma_{22}\, n_2^2 + 2\tau_{12}\, n_1 n_2$$

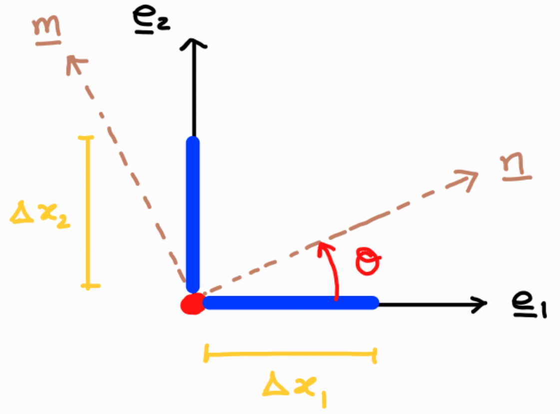
$$= \sigma_{11}\cos^2\theta + \sigma_{22}\sin^2\theta + 2\tau_{12}\sin\theta\cos\theta$$

$$\sigma_{nn} = \frac{\sigma_{11}+\sigma_{22}}{2} + \frac{\sigma_{11}-\sigma_{22}}{2}\cos 2\theta + \tau_{12}\sin 2\theta$$

$$\tau_{nn} = -\frac{\sigma_{11}-\sigma_{22}}{2}\sin 2\theta + \tau_{12}\cos 2\theta$$

Similarly, for plane strain case $[\underline{\underline{\epsilon}}] = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} \\ \epsilon_{12} & \epsilon_{22} \end{bmatrix}$, $\begin{bmatrix} \epsilon_{33}=0 \\ \epsilon_{13}=0 \\ \epsilon_{23}=0 \end{bmatrix}$

we can obtain similar relations for normal & shear strains at an arbitrary angle θ.



How are ϵ_{nn}, ϵ_{mm}, ϵ_{nm} related to ϵ_{11}, ϵ_{22}, ϵ_{12} ?

$$\epsilon_{nn} = \epsilon_{11}\cos^2\theta + \epsilon_{22}\sin^2\theta + 2\epsilon_{12}\sin\theta\cos\theta$$

$$= \frac{\epsilon_{11}+\epsilon_{22}}{2} + \frac{\epsilon_{11}-\epsilon_{22}}{2}\cos 2\theta + \epsilon_{12}\sin 2\theta$$

$$\epsilon_{mm} = \epsilon_{11}\cos^2\left(\frac{\pi}{2}+\theta\right) + \epsilon_{22}\sin^2\left(\frac{\pi}{2}+\theta\right) + 2\epsilon_{12}\sin\left(\frac{\pi}{2}+\theta\right)\cos\left(\frac{\pi}{2}+\theta\right)$$

$$= \epsilon_{11}\sin^2\theta + \epsilon_{22}\cos^2\theta - \epsilon_{12}\sin 2\theta$$

$$= \frac{\epsilon_{11}+\epsilon_{22}}{2} - \frac{\epsilon_{11}-\epsilon_{22}}{2}\cos 2\theta - \epsilon_{12}\sin 2\theta$$

$$\epsilon_{nm} = -\frac{\epsilon_{11}-\epsilon_{22}}{2}\sin 2\theta + \epsilon_{12}\cos 2\theta$$

Principal strains and principal directions

Like principal stresses, we can define principal strains. However, unlike principal stress planes, here we donot have principal strain planes. Instead, we only have principal strain directions.

Principal stress planes are planes on which the normal component of traction is maximized or minimized and no shear stress acts. The value of the normal component of traction on these planes are called principal stress components.

Similarly, out of the numerous line elements at a point in the body, the directions of those line elements that experience maximum / minimum normal strain are called principal strain directions.

The values of the normal strains in these directions are called principal strain components

Principal strains and principal directions

Like principal stresses, we can define principal strains

However, unlike principal stress planes, here we donot have principal strain planes. Instead, we have principal directions.

We know that at a point, principal stress planes are planes on which the normal component of traction is maximized or minimized. The value of the normal component of traction on these planes are principal stress components. Similarly, out of the numerous line elements at a point in the body, the directions of those line elements that experience maximum/minimum normal strain are called principal strain directions.

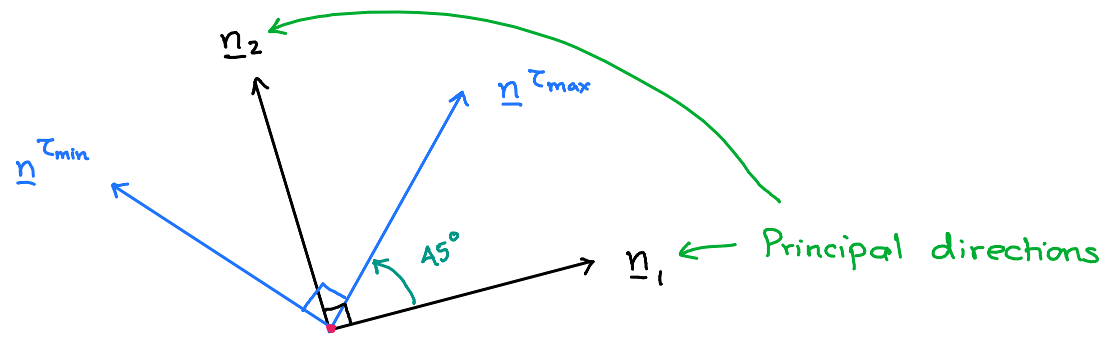
The values of the normal strains in these directions are called principal strain components. To find them, we do the same as earlier, i.e. we obtain eigenvectors and eigenvalues of the strain tensor

$$\underset{\approx}{\epsilon}\, \underset{\sim}{n} = \lambda\, \underset{\sim}{n}$$

The strain matrix in the coordinate system of principal strain directions becomes diagonal. As the off-diagonal elements will be zero, this means if we take two line elements directed along the principal strain directions, there will not be any change in angle between them.

Maximum Shear strain

We can also maximize shear strain at a point just like we maximized the shear component of traction. We had found that the planes on which shear stress becomes max/min lie at an angle of 45° from the principal planes. Similarly, the pair of perpendicular line elements that undergo maximum change in angle (or max shear strain) will be directed at 45° from principal strain directions.

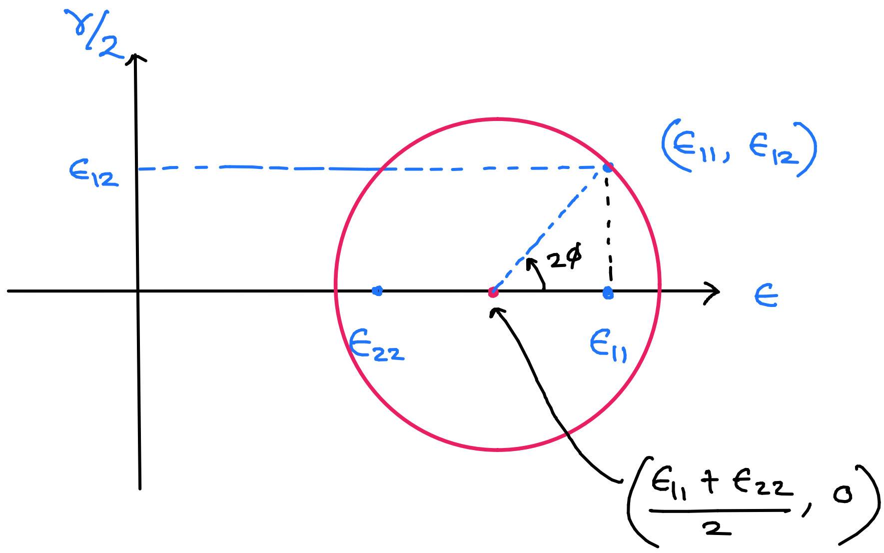


Mohr's Circle

Mohr's circle for stress gave the value of normal stress (σ) and shear stress (τ) on any arbitrary plane. Similarly, if we know the normal strain along two perpendicular directions say $\underline{e}_1$ and $\underline{e}_2$ and also know the shear strain between $\underline{e}_1$ and $\underline{e}_2$, then we can use Mohr's circle for strain to obtain normal and shear strain for two perpendicular line elements which are at an angle θ relative to $\underline{e}_1$ and $\underline{e}_2$ pair.

For 2D Mohr's circle, we need a state of strain s.t. at least one coordinate axis is along a principal direction

$$[\underline{\underline{\epsilon}}] = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & 0 \\ \epsilon_{12} & \epsilon_{22} & 0 \\ 0 & 0 & \epsilon_{33} \end{bmatrix}$$



Strain Invariants

Just like we have invariants of stress tensor denoted by $I_1(\underline{\underline{\sigma}})$, $I_2(\underline{\underline{\sigma}})$, $I_3(\underline{\underline{\sigma}})$, we have invariants of strain tensor denoted by $J_1(\underline{\underline{\epsilon}})$, $J_2(\underline{\underline{\epsilon}})$, $J_3(\underline{\underline{\epsilon}})$

$$J_1(\underline{\underline{\epsilon}}) = \epsilon_{11} + \epsilon_{22} + \epsilon_{33}$$

$$J_2(\underline{\underline{\epsilon}}) = \epsilon_{11}\epsilon_{22} + \epsilon_{22}\epsilon_{33} + \epsilon_{33}\epsilon_{11} - \epsilon_{12}^2 - \epsilon_{13}^2 - \epsilon_{23}^2$$

$$J_3(\underline{\underline{\epsilon}}) = \det([\underline{\underline{\epsilon}}])$$

Decomposition of strain tensor

We had seen the decomposition of stress tensor into hydrostatic and deviatoric parts. We can also decompose the strain tensor into two parts in a similar way

$$\underline{\underline{\epsilon}} = \underbrace{\frac{1}{3} J_1(\underline{\underline{\epsilon}})\, \underline{\underline{I}}}_{\text{volumetric/ spherical strain tensor}} + \underbrace{\left(\underline{\underline{\epsilon}} - \frac{1}{3} J_1(\underline{\underline{\epsilon}})\, \underline{\underline{I}} \right)}_{\text{deviatoric strain tensor}}$$

The first part is proportional to identity matrix. It is similar to the hydrostatic part of stress. This part called volumetric (or spherical) strain tensor is responsible for volume change and does not affect the shape of the body. The deviatoric part is responsible for distorting the body and changing its shape. The trace of the deviatoric strain tensor is zero.