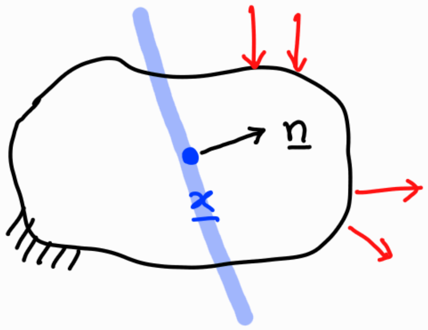
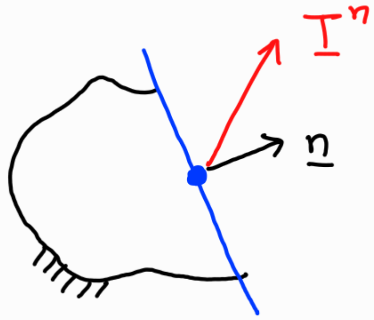
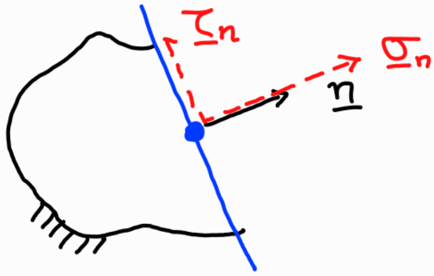


$\Rightarrow$



decompose $\underline{T}^n$ into
normal & shear components



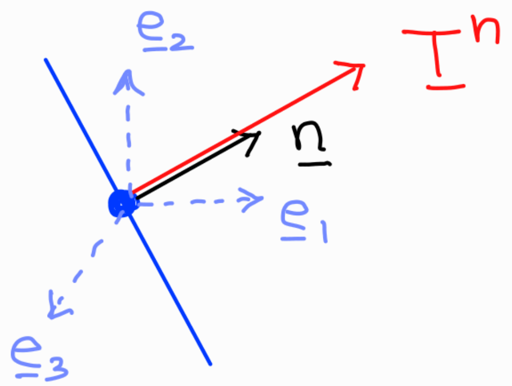
From failure considerations of materials, it is of interest to know the following:

(a) If there are any planes passing through a given point on which traction vector is wholly normal?
(i.e. traction vector has zero shear component & non-zero normal component)

(b) On which plane does the normal stress become maximum? What will be its magnitude?

(c) On which plane does the shear stress become maximum? What will be its magnitude?

Consider a plane with normal $\underline{n}$ s.t. the traction vector is oriented along the normal vector



$$[\underline{T}^n] = \lambda [\underline{n}] \quad \text{— (1)}$$

$$[\underline{T}^n] = [\underline{\underline{\sigma}}][\underline{n}] \quad \text{— (2)}$$

$$[\underline{\underline{\sigma}}][\underline{n}] = \lambda [\underline{n}]$$

$$\Rightarrow \quad [\underline{\underline{\sigma}} - \lambda \underline{\underline{I}}][\underline{n}] = \underline{0}$$

$$\Rightarrow \begin{bmatrix} \sigma_{11}-\lambda & \tau_{21} & \tau_{31} \\ \tau_{21} & \sigma_{22}-\lambda & \tau_{32} \\ \tau_{31} & \tau_{32} & \sigma_{33}-\lambda \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

A trivial soln is going to be $n_1 = n_2 = n_3 = 0$

For non-trivial soln, determinant must vanish

$$\lambda^3 - (\sigma_{11}+\sigma_{22}+\sigma_{33})\lambda^2 + (\sigma_{11}\sigma_{22}+\sigma_{22}\sigma_{33}+\sigma_{11}\sigma_{33} - \tau_{21}^2 - \tau_{31}^2 - \tau_{32}^2)\lambda$$

$$- (\sigma_{11}\sigma_{22}\sigma_{33} + 2\tau_{21}\tau_{32}\tau_{31} - \sigma_{11}\tau_{32}^2 - \sigma_{22}\tau_{31}^2 - \sigma_{33}\tau_{21}^2) = 0$$

Cubic eqn. → three roots

$\left.\lambda_1, \; \lambda_2, \; \lambda_3 \right\}$ 3 eigenvalues

Substitute $\lambda = \lambda_1$ $\Rightarrow$ $[\underline{n}] = [\underline{n}_1] = \begin{bmatrix} n_{11} \\ n_{21} \\ n_{31} \end{bmatrix}$ ← eigenvector

1st Principal stress

$$\begin{bmatrix} \sigma_{11}-\lambda_1 & \tau_{21} & \tau_{31} \\ & \sigma_{22}-\lambda_1 & \tau_{32} \\ \text{sym} & & \sigma_{33}-\lambda_1 \end{bmatrix} \underbrace{\begin{bmatrix} n_{11} \\ n_{21} \\ n_{31} \end{bmatrix}}_{[\underline{n}_1]} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$n_{11}^2 + n_{21}^2 + n_{31}^2 = 1$ ← the magnitude of the eigenvector is taken as 1

$[\underline{n}_1]$ → is the outward normal of the 1st principal plane

Substitute $\lambda = \lambda_2$ $\Rightarrow$ $[\underline{n}] = [\underline{n}_2] = \begin{bmatrix} n_{12} \\ n_{22} \\ n_{32} \end{bmatrix}$ ← eigenvector 2

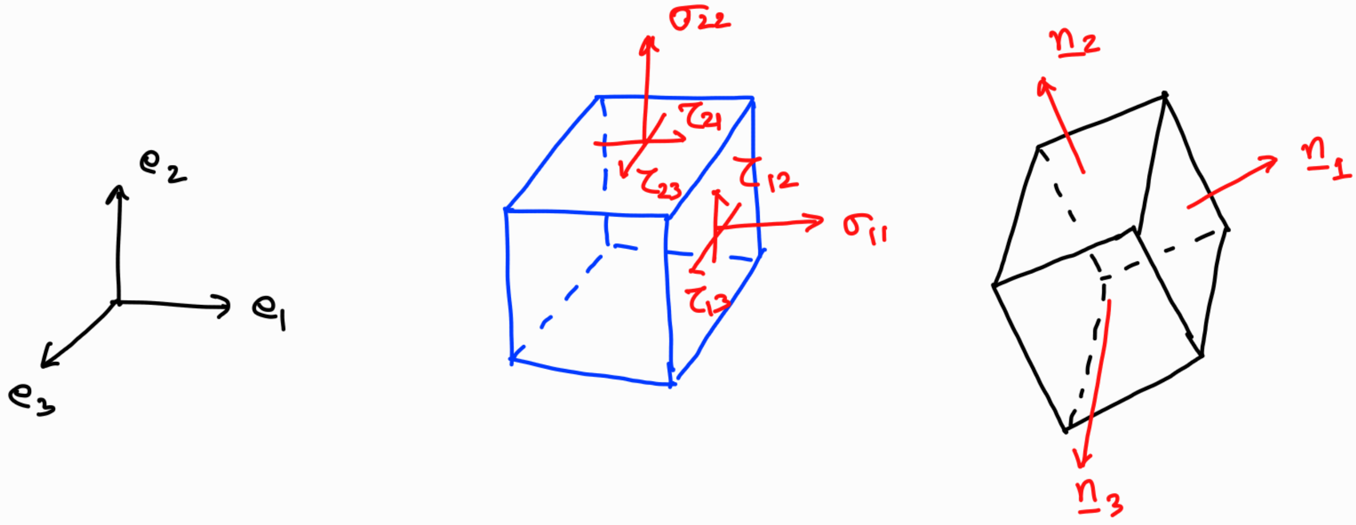
1st Principal stress

$$\begin{bmatrix} \sigma_{11}-\lambda_2 & \tau_{21} & \tau_{31} \\ & \sigma_{22}-\lambda_2 & \tau_{32} \\ \text{sym} & & \sigma_{33}-\lambda_2 \end{bmatrix} \underbrace{\begin{bmatrix} n_{12} \\ n_{22} \\ n_{32} \end{bmatrix}}_{[\underline{n}_2]} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$n_{12}^2 + n_{22}^2 + n_{32}^2 = 1$ ← the magnitude of the eigenvector is taken as 1

$[\underline{n}_2]$ → is the outward normal of the 2nd principal plane

Properties of principal planes at a point



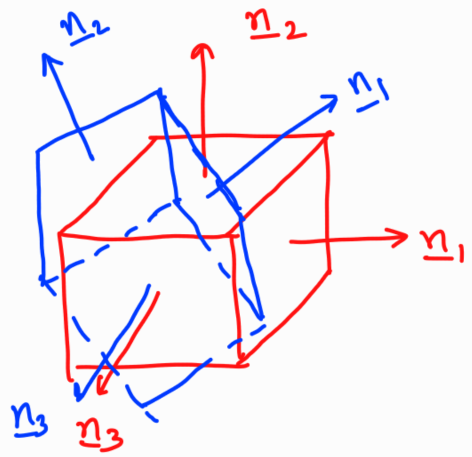
1) If eigenvalues $\lambda_1 \neq \lambda_2 \neq \lambda_3$ (distinct)

$$\underline{n}_1 \perp \underline{n}_2 \perp \underline{n}_3$$

2) $\lambda_1 = \lambda_2 \neq \lambda_3$ (two eigenvalues repeated)

$\downarrow$

$\underline{n}_3$ is unique



3) $\lambda_1 = \lambda_2 = \lambda_3$ (all eigenvalues repeated)

Every direction is a principal direction

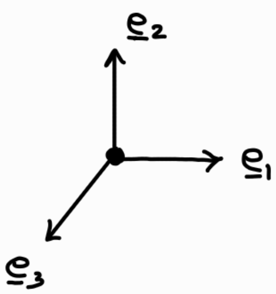
Representation of stress tensor in the coordinate system of its eigenvectors

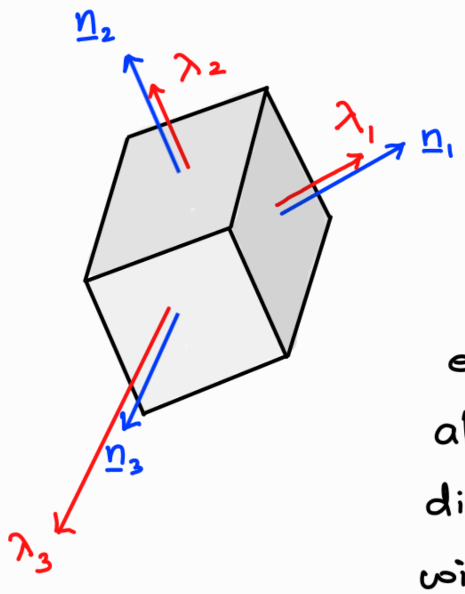
If you choose three mutually perpendicular eigenvectors to be the basis vectors of a coordinate system and then represent the stress tensor in the coordinate system

By definition, the traction on the principal planes will be $\lambda \underline{n}$ (no shear components would be present)

The stress matrix will be diagonal when expressed in the coordinate system spanned by principal directions

$$[\underline{\sigma}]_{\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$





With a cuboid element's faces along the principal directions, there will be no shear components

From failure considerations of materials, it is of interest to know the following:

(a) If there are any planes passing through a given point on which traction vector is wholly normal?
(i.e. traction vector has zero shear component & non-zero normal component) ✓ Principal planes/directions

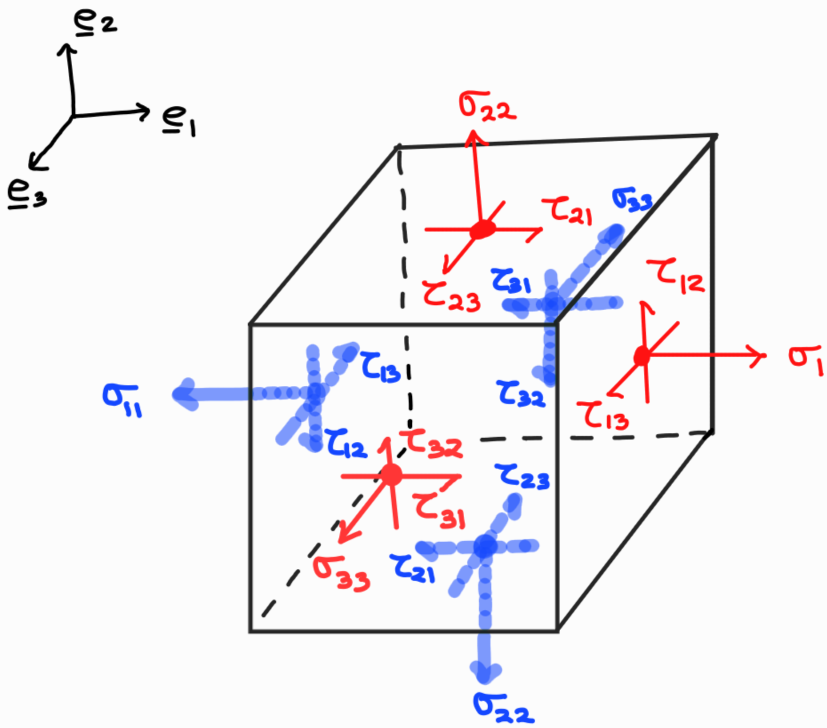
(b) On which plane does the normal stress become maximum? What will be its magnitude?

(c) On which plane does the shear stress become maximum? What will be its magnitude?

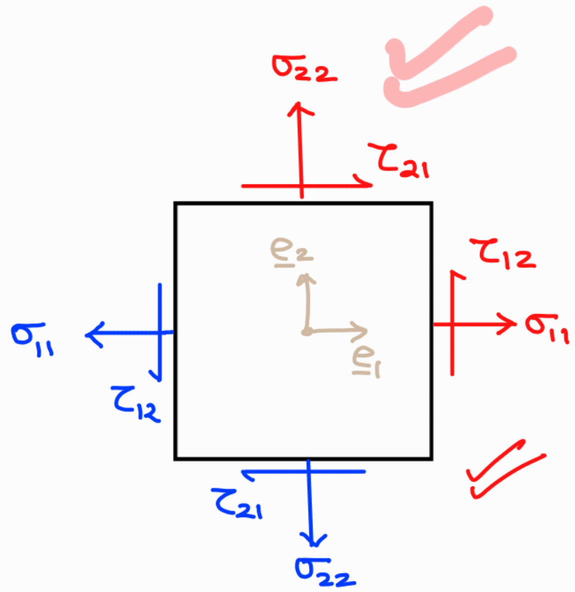
Turbine blades are subjected to various loads. It is important to know the maximum stress and its direction for proper design

We will now look at 2D plane stress case (instead of 3D) as this condition is most commonly assumed in practice.

At the end, we will discuss a method for finding the absolute maximum normal & shear stress at a point when the material is subject to both plane & 3D states of stress



3D state of stress

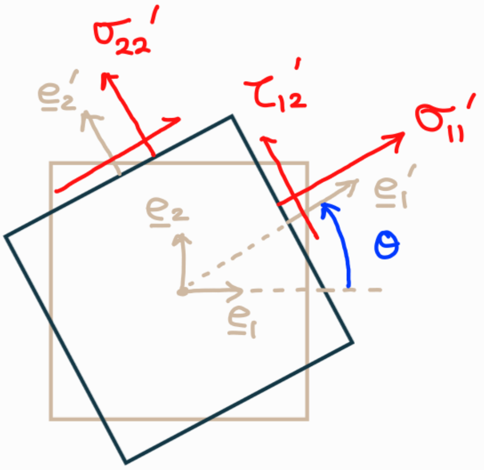


2D/plane state of stress

The general state of PLANE STRESS at a point is represented by two normal stress components, σ_{11} and σ_{22}, and one shear stress component, τ_{12}.

The values of these components will be different for different orientation of the plane stress element.

That is to say, if we rotate the plane stress element by an angle θ (say counterclockwise), then the stress component values will change to σ_{11}', σ_{22}', and τ_{12}'

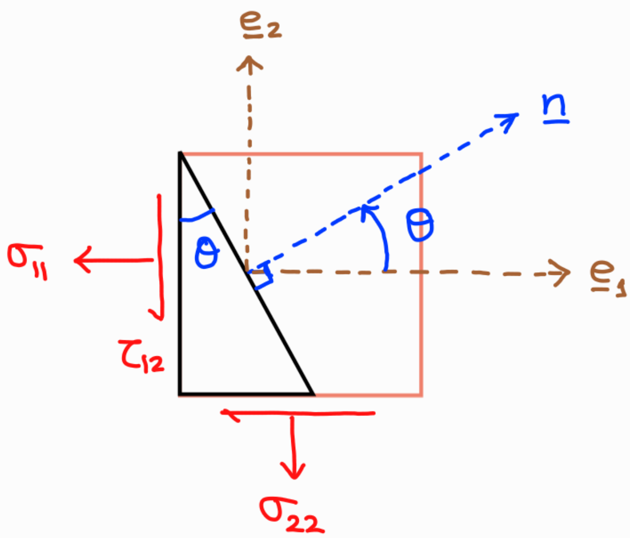


How are these plane stress components σ_{11}', σ_{22}', τ_{12}' related to σ_{11}, σ_{22}, τ_{12} via orientation θ?

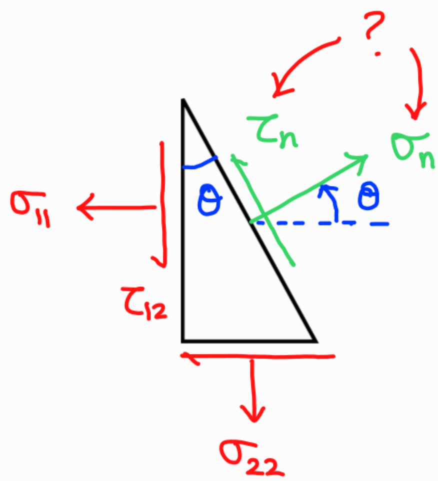
Can we find a θ for which normal stresses σ_{11}', σ_{22}' become max/min?

Can we find a θ for which stress stress τ_{12}' becomes absolute maximum?

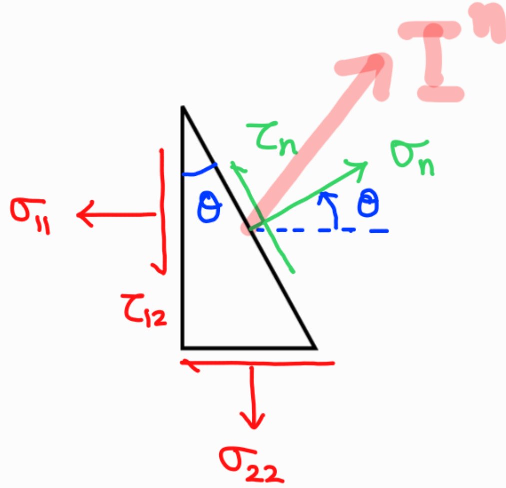
Plane Stress Transformation



$\Rightarrow$

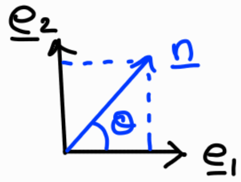


Traction on $\underline{n}$-plane at that point



$$\underline{T}^n = \underline{\underline{\sigma}}\,\underline{n}$$

Normal component of traction



$$\sigma_n = \underline{T}^n \cdot \underline{n}$$

$$= (\underline{\underline{\sigma}} \cdot \underline{n}) \cdot \underline{n}$$

$$= [\underline{n}]^T\,[\underline{\underline{\sigma}}]\,[\underline{n}]$$

$$= \begin{bmatrix} n_1 & n_2 \end{bmatrix} \begin{bmatrix} \sigma_{11} & \tau_{12} \\ \tau_{12} & \sigma_{22} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$$

$$= \sigma_{11} n_1^2 + \sigma_{22} n_2^2 + 2\tau_{12} n_1 n_2$$

$$= \sigma_{11} \cos^2\theta + \sigma_{22} \sin^2\theta + 2\tau_{12} \cos\theta \sin\theta$$

$$\boxed{\sigma_n(\theta) = \sigma_{11}\left(\frac{1+\cos 2\theta}{2}\right) + \sigma_{22}\left(\frac{1-\cos 2\theta}{2}\right) + \tau_{12}\sin 2\theta}$$

Find max/min $\sigma_n(\theta)$

$$\frac{\partial \sigma_n(\theta)}{\partial \theta} = 0 \longrightarrow \text{extreme points}$$

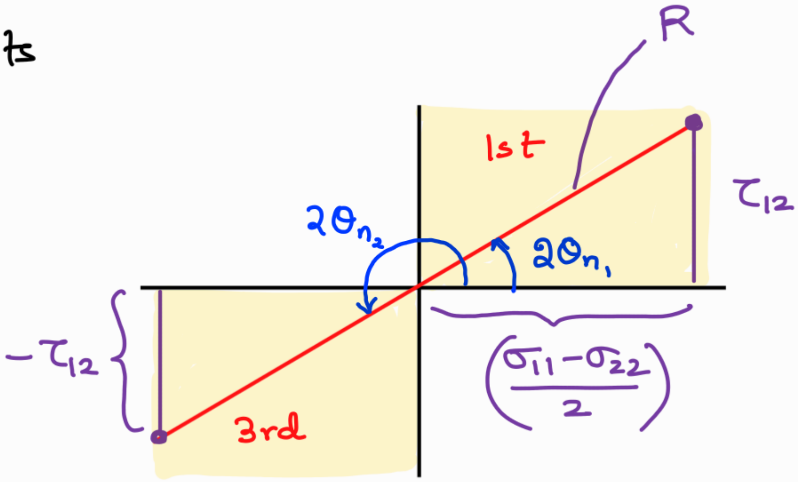
$$\frac{\partial \sigma_n}{\partial \theta} = -\frac{\sigma_{11}-\sigma_{22}}{2}(2\sin 2\theta) + 2\tau_{12}\cos 2\theta = 0$$

$$\tan 2\theta = \frac{\tau_{12}}{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)}$$

+ve in 1st & 3rd quadrants

1st $$2\theta_{n_1} = \frac{\tau_{12}}{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)}$$

3rd $$2\theta_{n_2} = \frac{-\tau_{12}}{-\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)}$$



$$R = \sqrt{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \tau_{12}^2}$$

The two solutions $2\theta_{n_1}$ and $2\theta_{n_2}$ are $180°$ apart, so in physical space, θ_{n_1} and θ_{n_2} are $90°$ apart.

To obtain maximum and minimum normal stress, we must substitute the angles θ_{n_1} and θ_{n_2} respectively.

$$\sigma_n(\theta) = \sigma_{11}\left(\frac{1 + \cos 2\theta}{2}\right) + \sigma_{22}\left(\frac{1 - \cos 2\theta}{2}\right) + \tau_{12} \sin 2\theta$$

$$= \frac{\sigma_{11} + \sigma_{22}}{2} + \frac{\sigma_{11} - \sigma_{22}}{2} \cos 2\theta + \tau_{12} \sin 2\theta$$

$\theta = \theta_{n_1} \; (\theta_{n_2})$

$$\cos 2\theta_{n_1, n_2} = \pm \frac{\sigma_{11} - \sigma_{22}}{2R}, \qquad \sin 2\theta_{n_1, n_2} = \pm \frac{\tau_{12}}{R}$$

$\sigma_{n_1} \leftarrow$ max normal stress, $\sigma_{n_2} \leftarrow$ min normal stress

$\theta_{n_1} \leftarrow$ corresponding angle (or normal direction) $\theta_{n_2} \leftarrow$ corresponding angle (or normal direction)

$$\sigma_{n_1, n_2} = \frac{\sigma_{11} + \sigma_{22}}{2} \pm \frac{\sigma_{11} - \sigma_{22}}{2}\left(\frac{\sigma_{11} - \sigma_{22}}{2R}\right) \pm \tau_{12}\left(\frac{\tau_{12}}{R}\right)$$

$$= \frac{\sigma_{11} + \sigma_{22}}{2} \pm \left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 \frac{1}{R} \pm \frac{\tau_{12}^2}{R}$$

$$= \frac{\sigma_{11} + \sigma_{22}}{2} \pm \frac{1}{R}\left[\underbrace{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \tau_{12}^2}_{R^2}\right]$$

$$R = \sqrt{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \tau_{12}^2}$$

$$\sigma_{n_1} = \frac{\cancelto{\lambda_1}{\sigma_{11}} + \cancelto{\lambda_2}{\sigma_{22}}}{2} + \sqrt{\left(\frac{\cancelto{\lambda_1}{\sigma_{11}} - \cancelto{\lambda_2}{\sigma_{22}}}{2}\right)^2 + \cancelto{0}{\tau_{12}}^2} \quad \text{max}$$

$$\sigma_{n_2} = \frac{\sigma_{11} + \sigma_{22}}{2} - \sqrt{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \tau_{12}^2} \quad \text{min}$$

$$\sigma_{n_1} = \frac{\lambda_1 + \lambda_2}{2} + \sqrt{\frac{(\lambda_1 - \lambda_2)^2}{2}} = \frac{\lambda_1 + \lambda_2}{2} + \frac{\lambda_1 - \lambda_2}{2}$$

$$= \lambda_1$$

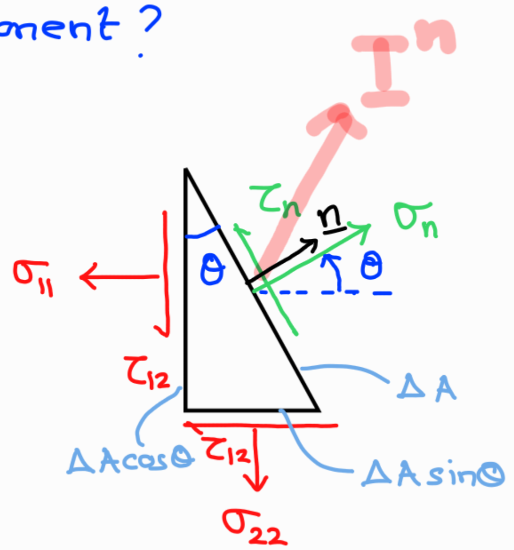
$$\sigma_{n_2} = \frac{\lambda_1 + \lambda_2}{2} - \frac{\lambda_1 - \lambda_2}{2} = \lambda_2$$

Let us also check what are the shear stresses on planes where normal stress components are maximum or minimum

For this, we can put the value of sin 2Θ and cos 2Θ in the relation for shear stress component τ_n :

How to derive the shear stress component?

Method 1: Equilibrium of forces along the inclined plane



$$+\nwarrow \sum F_{\underline{n}\perp} = 0$$

$$\Rightarrow \tau_n \Delta A + (\tau_{12}\Delta A \sin\theta)\sin\theta - (\sigma_{22}\Delta A\sin\theta)\cos\theta + (\sigma_{11}\Delta A\cos\theta)\sin\theta - (\tau_{12}\Delta A\cos\theta)\cos\theta = 0$$

$$\Rightarrow \tau_n = (\sigma_{22} - \sigma_{11})\sin\theta\cos\theta + \tau_{12}(\cos^2\theta - \sin^2\theta)$$

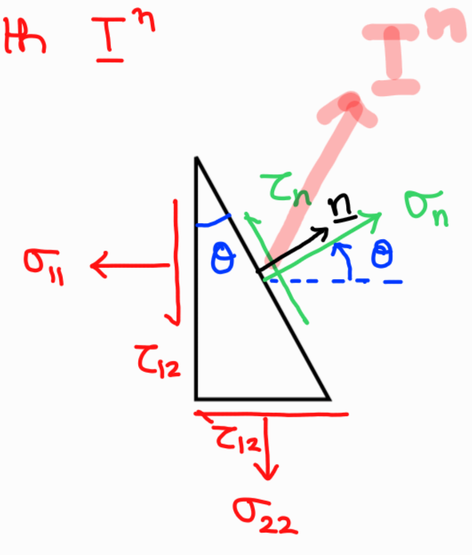
$$= -\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)\sin 2\theta + \tau_{12}\cos 2\theta$$

Method 2 : Dot product of $\underline{n}^{\perp}$ with $\underline{T}^n$

$$\tau_n = \underline{T}^n \cdot \underline{n}^{\perp}$$

$$= ([\underline{\underline{\sigma}}][\underline{n}]) \cdot [\underline{n}^{\perp}]$$

$$= \left(\begin{bmatrix} \sigma_{11} & \tau_{12} \\ \tau_{12} & \sigma_{22} \end{bmatrix}\begin{bmatrix} n_1 \\ n_2 \end{bmatrix}\right)^T \begin{bmatrix} n_1^{\perp} \\ n_2^{\perp} \end{bmatrix}$$

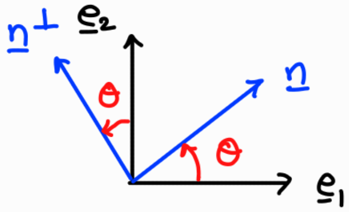


$$= \begin{bmatrix} n_1 & n_2 \end{bmatrix} \begin{bmatrix} \sigma_{11} & \tau_{12} \\ \tau_{12} & \sigma_{22} \end{bmatrix} \begin{bmatrix} n_1^{\perp} \\ n_2^{\perp} \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_{11} n_1 + \tau_{12} n_2 & \tau_{12} n_1 + \sigma_{22} n_2 \end{bmatrix} \begin{bmatrix} n_1^{\perp} \\ n_2^{\perp} \end{bmatrix}$$

$$= \sigma_{11} n_1 n_1^{\perp} + \tau_{12} n_2 n_1^{\perp} + \tau_{12} n_1 n_2^{\perp} + \sigma_{22} n_2 n_2^{\perp}$$

$$= \sigma_{11} n_1 n_1^{\perp} + \sigma_{22} n_2 n_2^{\perp} + \tau_{12} \left(n_1 n_2^{\perp} + n_2 n_1^{\perp} \right)$$



$$\underline{n} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \qquad \underline{n}^{\perp} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

$$= -\sigma_{11} \cos\theta \sin\theta + \sigma_{22} \sin\theta \cos\theta + \tau_{12} \left(\cos^2\theta - \sin^2\theta \right)$$

$$= -\left(\frac{\sigma_{11} - \sigma_{22}}{2} \right) \sin 2\theta + \tau_{12} \cos 2\theta$$

$$\sigma_n = \frac{\sigma_{11} + \sigma_{22}}{2} + \frac{\sigma_{11} - \sigma_{22}}{2} \cos 2\theta + \tau_{12} \sin 2\theta$$

$$\tau_n = - \frac{\sigma_{11} - \sigma_{22}}{2} \sin 2\theta + \tau_{12} \cos 2\theta$$

Let us also check what are the shear stresses on planes where normal stress components are maximum or minimum

$$\tau_n = - \left(\frac{\sigma_{11} - \sigma_{22}}{2} \right) \sin \overset{\left(\pm \frac{\tau_{12}}{R} \right)}{2\theta_{n_1, n_2}} + \tau_{12} \cos \overset{\pm \left(\frac{\sigma_{11} - \sigma_{22}}{2R} \right)}{2\theta_{n_1, n_2}}$$

$$= \mp \left(\frac{\sigma_{11} - \sigma_{22}}{2} \right) \left(\frac{\tau_{12}}{R} \right) \pm \tau_{12} \left(\frac{\sigma_{11} - \sigma_{22}}{2R} \right) = 0$$

- Therefore, we find that the shear stress components are zero on planes where normal stress components are maximized or minimized.

- Coincidentally principal planes are also planes where there are only normal stress components and zero shear stresses.

Thus, principal planes are also planes where the normal stresses are max/min and shear stresses are zero!

Back to our questions:

(a) If there are any planes passing through a given point on which traction vector is wholly normal?
(i.e. traction vector has zero shear component & non-zero normal component) Principal planes ✓

(b) On which plane does the normal stress become maximum? What will be its magnitude? Principal planes ✓
Principal stresses ✓

(c) On which plane does the shear stress become maximum? What will be its magnitude?

Maximum shear stress and corresponding planes

$$\tau_n(\theta) = -\left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)\sin 2\theta + \tau_{12}\cos 2\theta$$

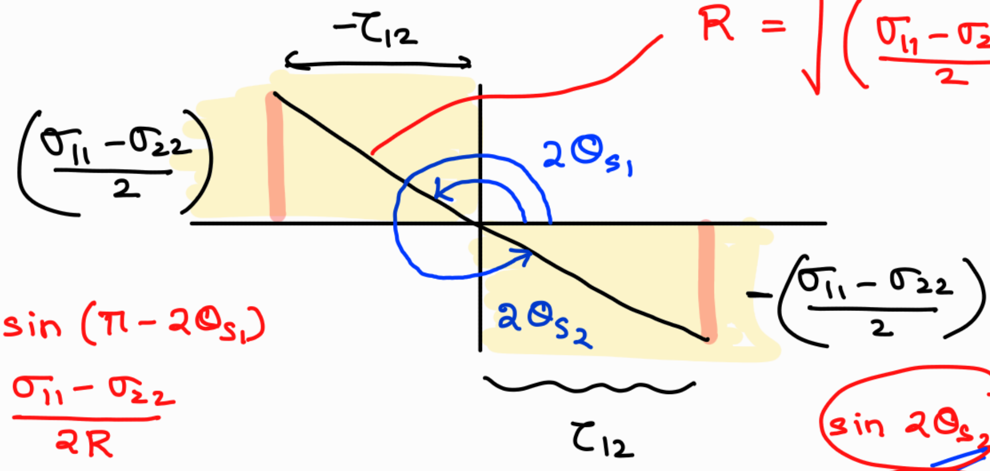
To find the maximum or minimum value of τ_n w.r.t θ,

$$\frac{\partial \tau_n}{\partial \theta} = 0 \Rightarrow \tan 2\theta = -\left(\frac{\sigma_{11}-\sigma_{22}}{2\tau_{12}}\right)$$

Assume $(\sigma_{11} > \sigma_{22})$

−ve in → 2nd quad
→ 4th quad

$$R = \sqrt{\left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)^2 + \tau_{12}^2}$$



$$\sin 2\theta_{s_1} = \sin(\pi - 2\theta_{s_1}) = \frac{\sigma_{11}-\sigma_{22}}{2R}$$

$$\cos 2\theta_{s_1} = -\cos(\pi - 2\theta_{s_1}) = -\frac{\tau_{12}}{R}$$

$$\sin 2\theta_{s_2} = -\frac{\sigma_{11}-\sigma_{22}}{2\tau_{12}}$$

$$\cos 2\theta_{s_2} = \frac{\tau_{12}}{R}$$

The two planes $2\theta_{s_1}$ and $2\theta_{s_2}$ are 180° apart, which implies, in the physical space they are 90° apart.

negative reciprocal

$$\tan 2\theta_n = \frac{\tau_{12}}{\left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)}, \quad \tan 2\theta_s = -\frac{\sigma_{11}-\sigma_{22}}{2\tau_{12}}$$

also principal plane angles/directions

$\theta_n \angle 45^\circ$ with θ_s

The max/min values of shear stress is obtained by putting the values of θ_s in the relation of $\tau_s(\theta)$

$$\tau_{n,\min} = -\left(\frac{\sigma_{11}-\sigma_{22}}{2}\right) \overset{\frac{\sigma_{11}-\sigma_{22}}{2R}}{\cancel{\sin 2\theta_{s_1}}} + \tau_{12}\, \overset{-\frac{\tau_{12}}{R}}{\cancel{\cos 2\theta_{s_1}}}$$

$$= -\left[\left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)^2 + \tau_{12}^2\right] / R$$

$$= -\sqrt{\left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)^2 + \tau_{12}^2} \quad \text{(min value)}$$

Similarly,

$$\tau_{n,\max} = -\left(\frac{\sigma_{11}-\sigma_{22}}{2}\right) \overset{-\frac{\sigma_{11}-\sigma_{22}}{2R}}{\cancel{\sin 2\theta_{s_2}}} + \tau_{12}\, \overset{\frac{\tau_{12}}{R}}{\cancel{\cos 2\theta_{s_2}}}$$

$$= +\left[\left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)^2 + \tau_{12}^2\right] / R$$

$$= +\sqrt{\left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)^2 + \tau_{12}^2} \quad \text{(max value)}$$

What would be the max/min values of shear stress at a point when expressed in principal coordinate sys?

$$\tau_{n,\substack{\max\\ \min}} = \pm\sqrt{\left(\frac{\lambda_1-\lambda_2}{2}\right)^2} = \pm\left|\frac{\lambda_1-\lambda_2}{2}\right|$$

What are the normal stresses on the planes of max/min shear stresses?

Set the values of θ_{s_1} and θ_{s_2} in the relation for $\sigma_n(\theta)$

$$\sigma_n(\theta=\theta_{s_1,s_2}) = \frac{\sigma_{11}+\sigma_{22}}{2} + \frac{\sigma_{11}-\sigma_{22}}{2}\cos 2\theta_s + \tau_{12}\sin 2\theta_s$$

$$= \frac{\sigma_{11}+\sigma_{22}}{2} + \frac{\sigma_{11}-\sigma_{22}}{2}\left(\mp\frac{\tau_{12}}{R}\right) + \tau_{12}\left(\pm\frac{\sigma_{11}-\sigma_{22}}{2R}\right)$$

$$= \frac{\sigma_{11}+\sigma_{22}}{2} \quad \text{(average normal stress)}$$

Stress element representation