Stress Tensor

$$\underline{T}^{n}(\underline{x}) = \underline{\underline{\sigma}}(\underline{x})\,\underline{n}$$

Stress tensor

The stress tensor depends on $\underline{x}$ alone.

To obtain the stress tensor at a point:

a) choose three mutually perpendicular planes at that point,

b) find tractions on those planes

c) get the components of the tractions on the planes

Stress matrix as a representation of stress tensor

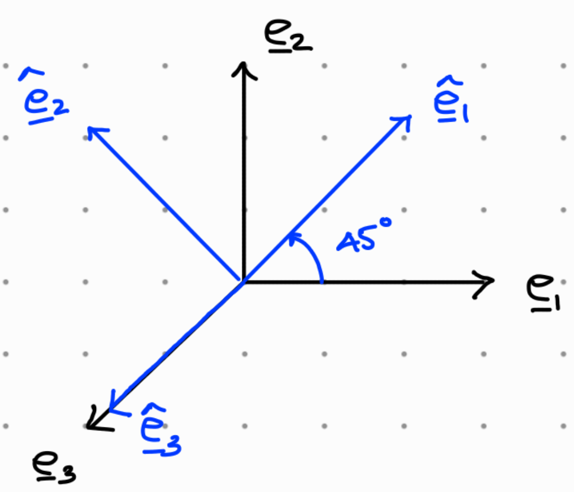
The state of stress at a point is completely defined by the NINE stress components acting on three mutually perpendicular planes (say $\underline{e}_1 - \underline{e}_2 - \underline{e}_3$ planes)

$$[\sigma]_{(\underline{e}_1, \underline{e}_2, \underline{e}_3)} = \begin{array}{ccc} \left[\begin{array}{ccc} \sigma_{11} & \tau_{21} & \tau_{31} \\ \tau_{12} & \sigma_{22} & \tau_{32} \\ \tau_{13} & \tau_{23} & \sigma_{33} \end{array}\right] \\ \begin{array}{ccc} \underline{T}^1 & \underline{T}^2 & \underline{T}^3 \end{array} \end{array}$$

Suppose the stress matrix in $(\underset{\sim}{e}_1 - \underset{\sim}{e}_2 - \underset{\sim}{e}_3)$ coordinate system is given by

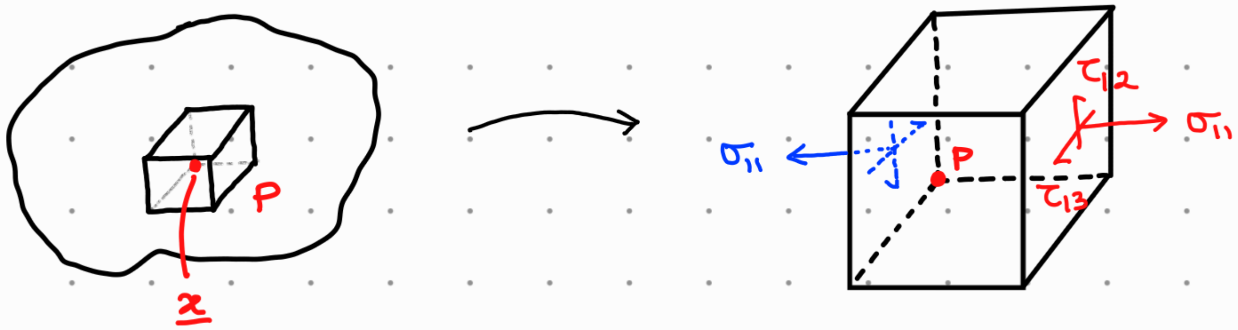
$$[\underset{\approx}{\sigma}]_{(e_1-e_2-e_3)} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

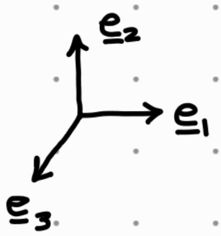
Think of a new coordinate system obtained by rotation of $\underset{\sim}{e}_1 - \underset{\sim}{e}_2 - \underset{\sim}{e}_3$ coordinate system by $45°$ about $\underset{\sim}{e}_3$. Find the stress matrix in the new coordinate system.



Cuboidal representation of stress tensor

To represent the stress tensor at a given point $\underline{x}$ in a body, we take an infinitesimally small cuboid





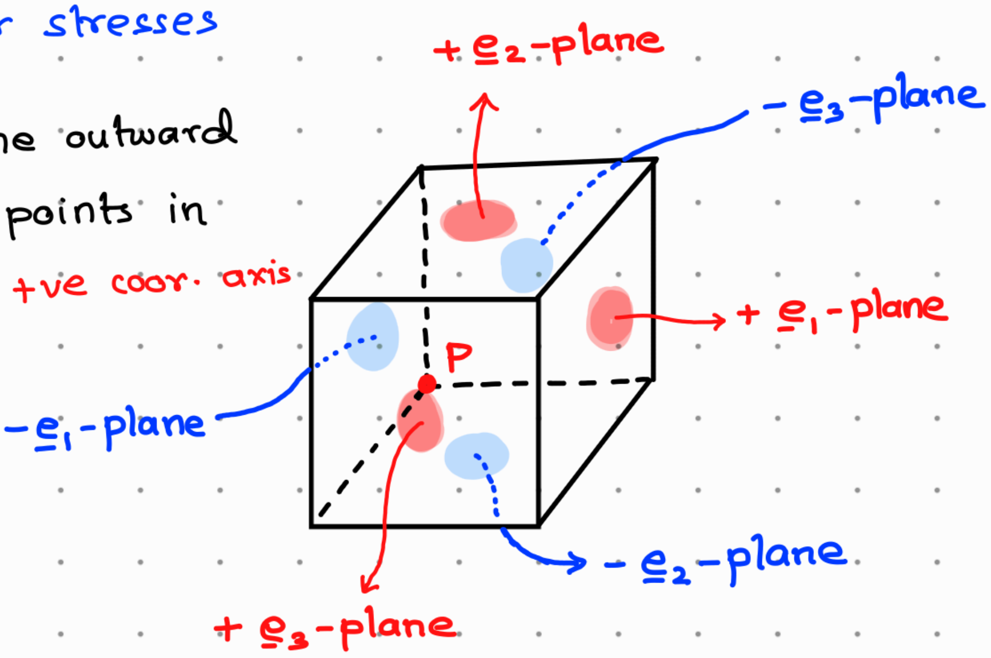
How do we define which stress is positive or negative?

→ We need a SIGN CONVENTION

Sign Convention for stresses

- A face is +ve if the outward face normal vector points in the direction of the +ve coor. axis

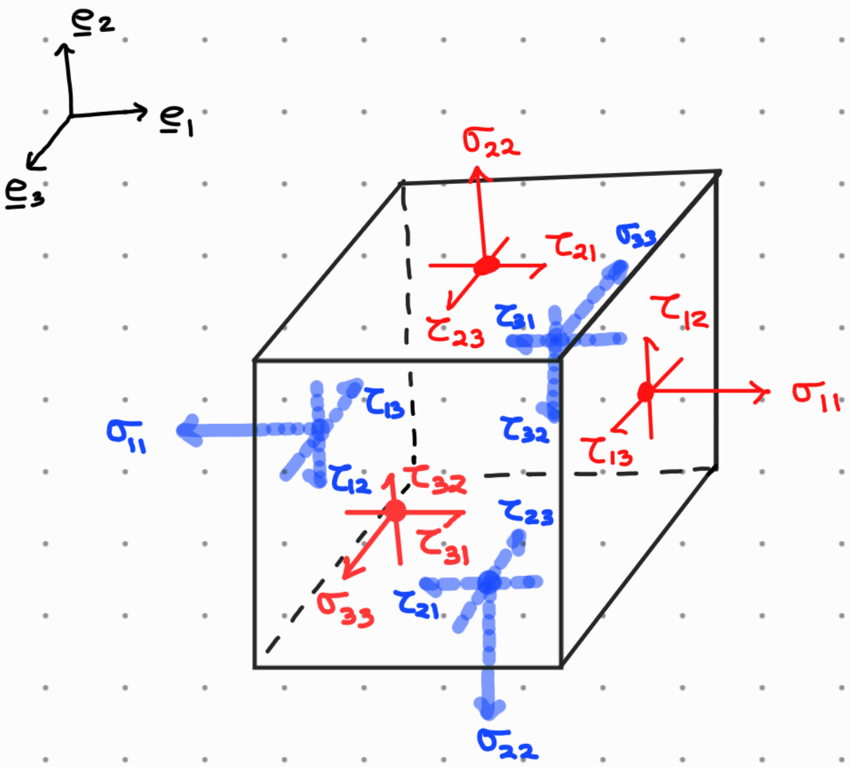
- A face is -ve if the outward face normal points in the direction of the -ve coordinate axis



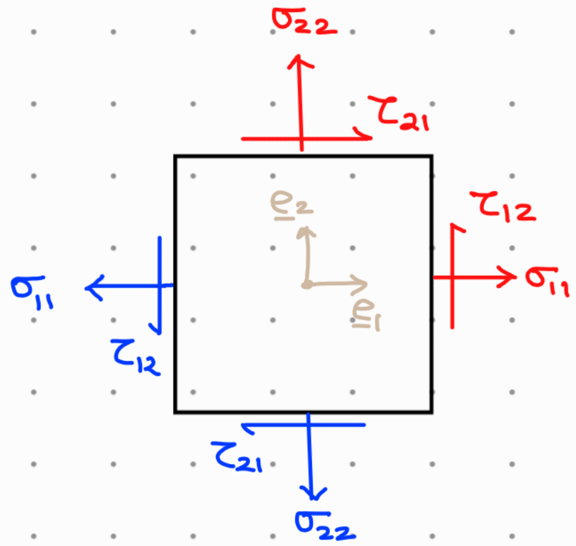
- The stress component is +ve when a positively directed force component acts on a positive face

- The stress component is +ve when a negatively directed force component acts on a negative face

- When a positively directed force component acts on a negative face or a negatively directed force component acts on a positive face, the stress component is negative



3D state of stress



2D/plane state of stress

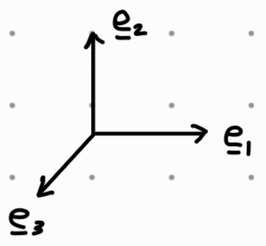
Certain remarks:

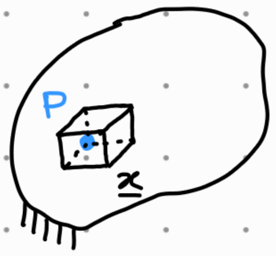
- The cuboid represents a very very small volume element from inside the body at the point P

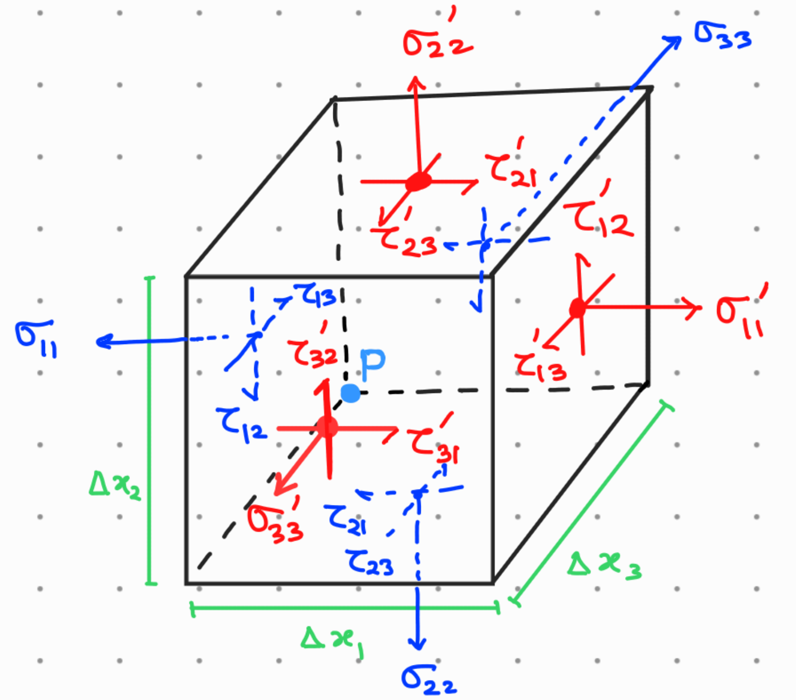
- Note that all stress components shown are positive
  - Positively directed force component act on +ve face
  - Negatively directed force component act on −ve face

- The stress components are assumed to be uniform over the face

Stress Equilibrium Equations in Cartesian Coordinates







The stress components on faces at distances Δx_1, Δx_2, and Δx_3 are different from the components at the faces passing through $P(\underline{x})$

How are σ_{11}', τ_{12}', τ_{13}' related to σ_{11}, τ_{12}, τ_{13} ?

Stress components on left face

$\sigma_{11} = \sigma_{11}(x_1, x_2, x_3)$

$\tau_{12} = \tau_{12}(x_1, x_2, x_3)$

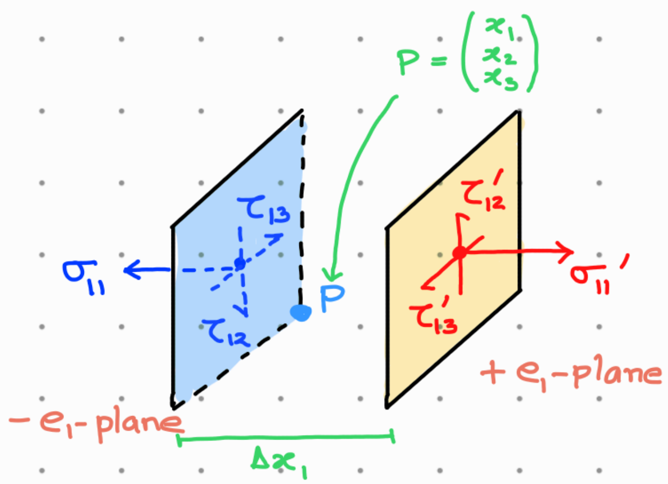
$\tau_{13} = \tau_{13}(x_1, x_2, x_3)$

Stress components on right face

$\sigma_{11}' = \sigma_{11}(x_1+\Delta x_1, x_2, x_3)$

$\tau_{12}' = \tau_{12}(x_1+\Delta x_1, x_2, x_3)$

$\tau_{13}' = \tau_{13}(x_1+\Delta x_1, x_2, x_3)$



Face areas: $\Delta x_2 \Delta x_3$

Using Taylor series expansion,

$$\sigma_{11}' \approx \sigma_{11}(x_1, x_2, x_3) + \frac{\partial \sigma_{11}}{\partial x_1}(x_1, x_2, x_3)\,\Delta x_1$$

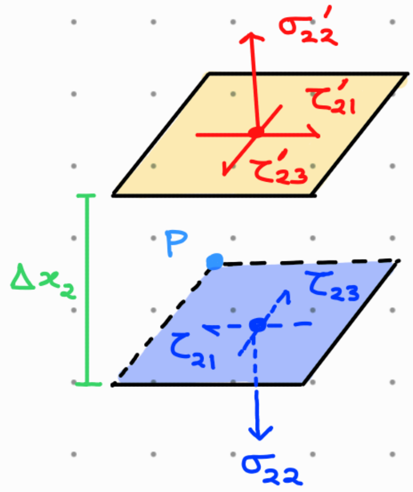
$$\tau_{12}' \approx \tau_{12}(x_1, x_2, x_3) + \frac{\partial \tau_{12}}{\partial x_1}(x_1, x_2, x_3)\,\Delta x_1$$

$$\tau_{13}' \approx \tau_{13}(x_1, x_2, x_3) + \frac{\partial \tau_{13}}{\partial x_1}(x_1, x_2, x_3)\,\Delta x_1$$

Similarly, we can express relationships for other faces

Face area: $\Delta x_1 \Delta x_3$

Stress components on bottom face



$$\sigma_{22} = \sigma_{22}(x_1, x_2, x_3)$$

$$\tau_{21} = \tau_{21}(x_1, x_2, x_3)$$

$$\tau_{23} = \tau_{23}(x_1, x_2, x_3)$$

Stress components on top face

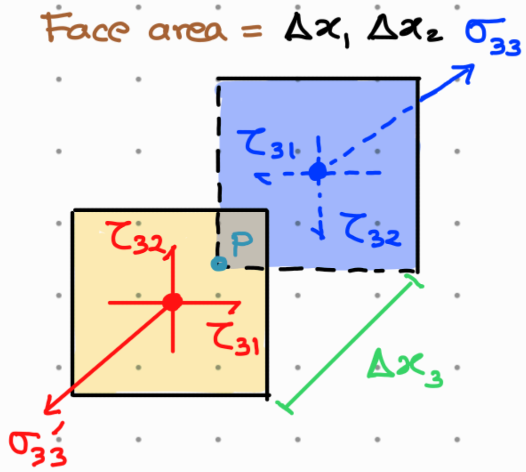
$$\sigma_{22}' = \sigma_{22}(x_1, x_2 + \Delta x_2, x_3) \approx \sigma_{22}(\underline{x}) + \frac{\partial \sigma_{22}(\underline{x})}{\partial x_2} \Delta x_2$$

$$\tau_{21}' = \tau_{21}(x_1, x_2 + \Delta x_2, x_3) \approx \tau_{21}(\underline{x}) + \frac{\partial \tau_{21}(\underline{x})}{\partial x_2} \Delta x_2$$

$$\tau_{23}' = \tau_{23}(x_1, x_2 + \Delta x_2, x_3) \approx \tau_{23}(\underline{x}) + \frac{\partial \tau_{23}(\underline{x})}{\partial x_2} \Delta x_2$$

Face area = $\Delta x_1 \Delta x_2$

Stress components on back face



$$\sigma_{33} = \sigma_{33}(x_1, x_2, x_3)$$

$$\tau_{31} = \tau_{31}(x_1, x_2, x_3)$$

$$\tau_{32} = \tau_{32}(x_1, x_2, x_3)$$

Stress components on front face

$$\sigma_{33}' = \sigma_{33}(x_1, x_2, x_3 + \Delta x_3) \approx \sigma_{33}(\underline{x}) + \frac{\partial \sigma_{33}(\underline{x})}{\partial x_3} \Delta x_3$$

$$\tau_{31}' = \tau_{31}(x_1, x_2, x_3 + \Delta x_3) \approx \tau_{31}(\underline{x}) + \frac{\partial \tau_{31}(\underline{x})}{\partial x_3} \Delta x_3$$

$$\tau_{32}' = \tau_{32}(x_1, x_2, x_3 + \Delta x_3) \approx \tau_{32}(\underline{x}) + \frac{\partial \tau_{32}(\underline{x})}{\partial x_3} \Delta x_3$$

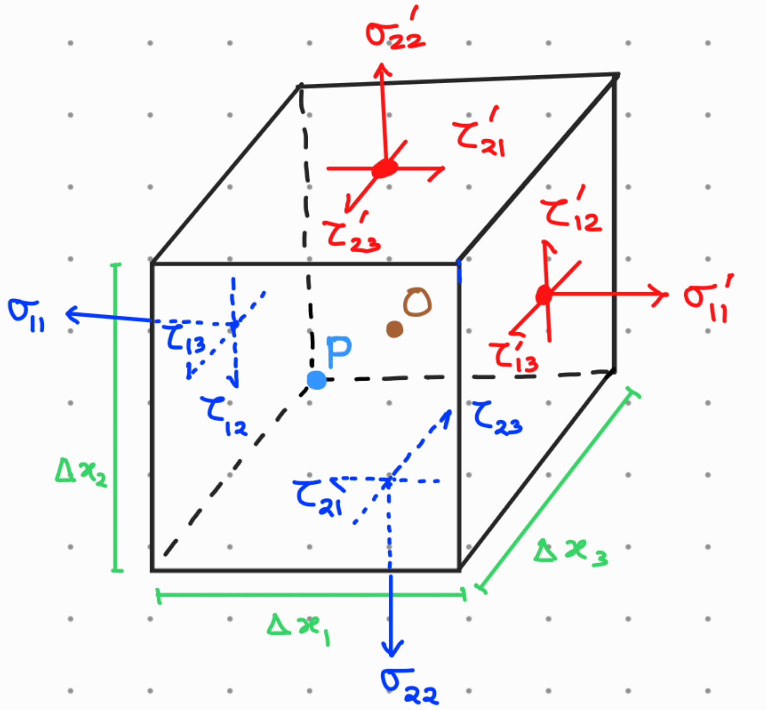
If a deformable body is in equilibrium, then any isolated part of the body must also be in equilibrium.

So, the cuboid must also be in equilibrium:

$$\sum \underline{M}_{\text{about any chosen pt}} = \underline{0} \Leftrightarrow \text{Balance of angular momentum}$$

$$\checkmark \sum \underline{F} = \underline{0} \Leftrightarrow \text{Balance of linear momentum}$$

Let's consider the moment about the center of cuboid O



About the $\underline{e}_3$-axis

$$+\circlearrowleft \sum M_o = 0$$

$$\Rightarrow (\tau'_{12}\,\Delta x_2\,\Delta x_3)\frac{\Delta x_1}{2} - (\tau'_{21}\,\Delta x_1\,\Delta x_3)\frac{\Delta x_2}{2} + (\tau_{12}\,\Delta x_2\,\Delta x_3)\frac{\Delta x_1}{2} - (\tau_{21}\,\Delta x_1\,\Delta x_3)\frac{\Delta x_2}{2} = 0$$

$$\Rightarrow \left[\tau_{12} + \frac{\partial \tau_{12}}{\partial x_1}\Delta x_1 + \tau_{12} - \tau_{21} - \frac{\partial \tau_{21}}{\partial x_2}\Delta x_2 - \tau_{21}\right]\frac{\Delta x_1\,\Delta x_2\,\Delta x_3}{2} = 0$$

$$\Rightarrow 2\tau_{12} - 2\tau_{21} + \frac{\partial \tau_{12}}{\partial x_1}\Delta x_1 - \frac{\partial \tau_{21}}{\partial x_2}\Delta x_2 = 0$$

As $\Delta x_1, \Delta x_2, \Delta x_3 \to 0$

$\boxed{\tau_{12} = \tau_{21}}$, Likewise, you can get

$\boxed{\tau_{13} = \tau_{31}}$, $\boxed{\tau_{23} = \tau_{32}}$

Shear stress components on perpendicular faces are equal in mag.

$$\underline{\gamma} = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix}$$

$$\xrightarrow{+} \sum F_x = 0$$

$$\cancel{\sigma_{11}} + \frac{\partial \sigma_{11}}{\partial x_1}\Delta x_1$$

$$\Rightarrow \cancel{\sigma_{11}'}\Delta x_2 \Delta x_3 - \cancel{\sigma_{11}\Delta x_2 \Delta x_3}$$

$$\cancel{\tau_{21}} + \frac{\partial \tau_{21}}{\partial x_2}\Delta x_2$$

$$+ \cancel{\tau_{21}'}\Delta x_1 \Delta x_3 - \cancel{\tau_{21}\Delta x_1 \Delta x_3}$$

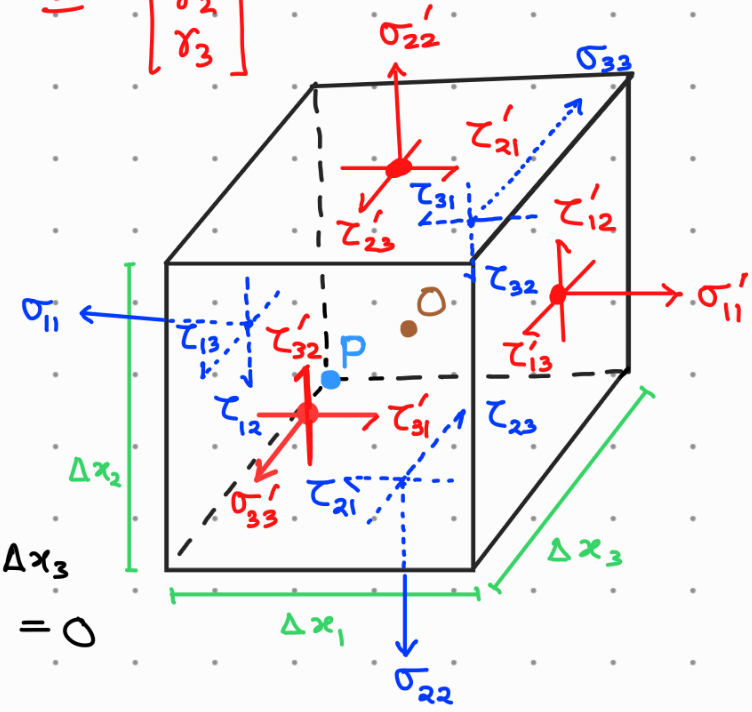
$$\cancel{\tau_{31}} + \frac{\partial \tau_{31}}{\partial x_3}\Delta x_3$$

$$+ \cancel{\tau_{31}'}\Delta x_1 \Delta x_2 - \cancel{\tau_{31}\Delta x_1 \Delta x_2}$$

$$+ \gamma_1 \Delta x_1 \Delta x_2 \Delta x_3 = 0$$

$$\Rightarrow \left[\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \tau_{21}}{\partial x_2} + \frac{\partial \tau_{31}}{\partial x_3} + \gamma_1\right]\Delta x_1 \Delta x_2 \Delta x_3 = 0$$

$$\Rightarrow \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \tau_{21}}{\partial x_2} + \frac{\partial \tau_{31}}{\partial x_3} + \gamma_1 = 0$$



Similarly,

$$+\uparrow \sum F_y = 0 \quad \Rightarrow \quad \frac{\partial \tau_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \tau_{32}}{\partial x_3} + \gamma_2 = 0$$

$$\overset{+}{\swarrow} \sum F_z = 0 \quad \Rightarrow \quad \frac{\partial \tau_{13}}{\partial x_1} + \frac{\partial \tau_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + \gamma_3 = 0$$

Stress Equilibrium Relations

Three force eq^n + Three moment eq^n $\Rightarrow$ Total SIX relations

Total NINE unknowns

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \tau_{21}}{\partial x_2} + \frac{\partial \tau_{31}}{\partial x_3} + \gamma_1 = 0 \qquad \tau_{21} = \tau_{12}$$

$$\frac{\partial \tau_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \tau_{32}}{\partial x_3} + \gamma_2 = 0 \qquad \tau_{23} = \tau_{32}$$

$$\frac{\partial \tau_{13}}{\partial x_1} + \frac{\partial \tau_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + \gamma_3 = 0 \qquad \tau_{31} = \tau_{13}$$

How many in 2D ?

In indicial notation, one can write

$$\sum_{i=1}^{3} \frac{\partial \sigma_{ij}(\underline{x})}{\partial x_i} + \gamma_j = 0$$

$$\underline{\underline{\sigma}}(\underline{x}) = \underline{\underline{\sigma}}^T(\underline{x})$$

We have six equilibrium equations, but nine unknown stress components

We need more equations which we will obtain from stress-strain relation & strain-displacement

The above equations are partial differential equations, and to solve them uniquely, we would need BOUNDARY CONDITIONS

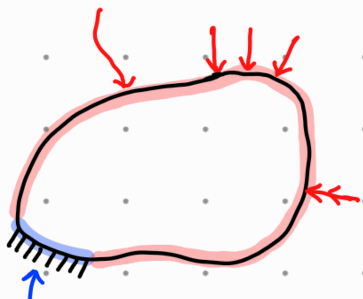
There are two types of boundary conditions

→ displacement BCs (Dirichlet BCs)

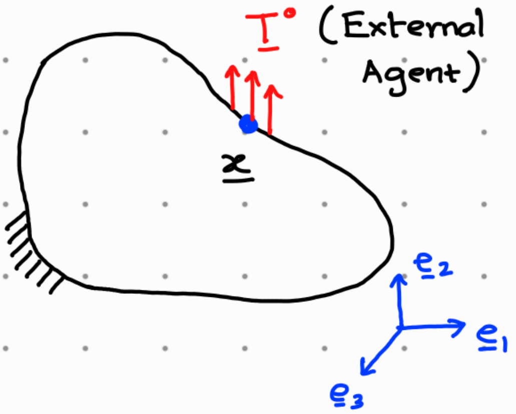
specifies displacements on the boundary (will cover later in deformation)

→ traction / force BCs (Neumann BCs)

specifies external loads applied on the boundary

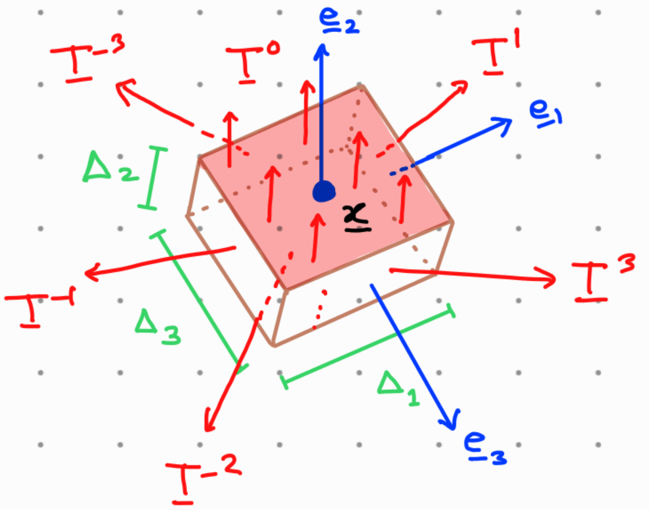


Derivation of traction boundary condition



This applied load is usually distributed over an area; therefore has unit of traction. We denote this distributed $\underline{T}^\circ$

Let's take a small piece from the surface. The actual boundary is curved but if we take a tiny piece, it will be almost flat



On the top surface of this small body part acts the external uniform distributed traction $\underline{T}^0$.

The directions of $\underline{e}_1 - \underline{e}_3$ are in the plane of the surface and $\underline{e}_2$ is the normal to the plane.

What are the forces acting on this small body part?

- External force $\underline{T}^0$ (acting on exposed surface)
- Tractions (acting on the internal faces)
- Body force (acting at the COM of the volume)

The total force on this small body part must be also be in equilibrium.

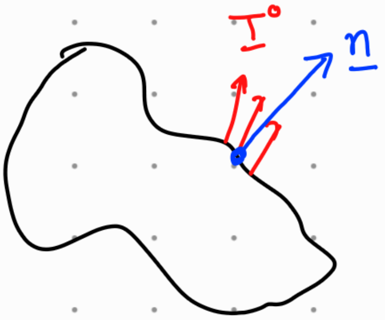
$$\underbrace{\underline{T}^0 \Delta_1 \Delta_3}_{\text{Ext force}} + (\underline{T}^1 + \underline{T}^{-1}) \Delta_2 \Delta_3 + (\underline{T}^3 + \underline{T}^{-3}) \Delta_1 \Delta_2 + \underline{T}^{-2} \Delta_1 \Delta_3 + \underbrace{\underline{\gamma} \Delta_1 \Delta_2 \Delta_3}_{\text{Body force}} = \underline{0}$$

If you divide the above equation by $\Delta_1 \Delta_3$, we get:

$$\Rightarrow \underline{T}^0 + (\underline{T}^1 + \underline{T}^{-1}) \frac{\cancelto{0}{\Delta_2}}{\Delta_1} + (\underline{T}^3 + \underline{T}^{-3}) \frac{\cancelto{0}{\Delta_2}}{\Delta_3} + \underline{T}^{-2} + \gamma \cancelto{0}{\Delta_2} = 0$$

$$\boxed{\underline{T}^0 = -\underline{T}^{-2}} \Rightarrow \underline{T}^0 = \underline{T}^2 \Rightarrow \underline{T}^0 = \underline{\underline{\sigma}}\, \underline{e}_2$$

Now, let $\Delta_2 \to 0$, that is we shrink the height by pushing the bottom surface towards the top surface while keeping the surface area $\Delta_1 \Delta_3$ constant.



$$\boxed{\underline{\underline{\sigma}}\,\underline{n} = \underline{T}^{\circ}}$$

General formula

$$\begin{bmatrix} \sigma_{11} & T_1^{\circ} & \tau_{31} \\ T_1^{\circ} & T_2^{\circ} & T_3^{\circ} \\ \tau_{31} & T_3^{\circ} & \sigma_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} T_1^{\circ} \\ T_2^{\circ} \\ T_3^{\circ} \end{bmatrix}$$

$$\underline{\underline{\sigma}} \qquad \underline{e}_2 \qquad \underline{T}^{\circ}$$