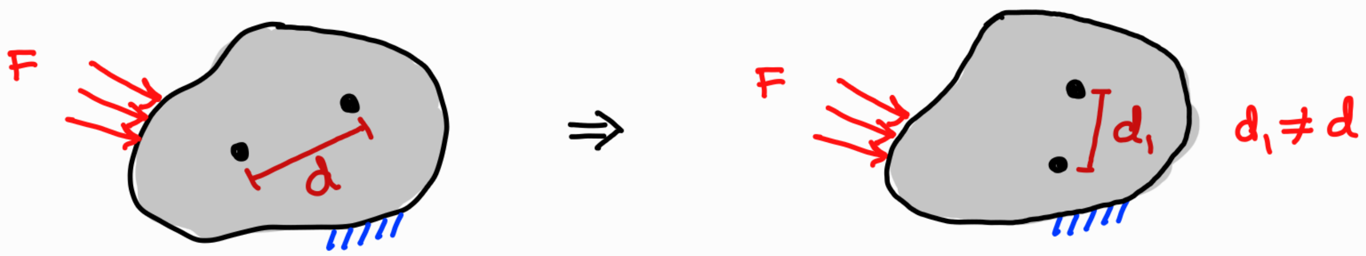
Rigid body → Distance between two points remain same before and after applying external forces/moments.

This is not true for DEFORMABLE bodies.



Therefore, in the analysis of deformable bodies we need additional conditions than just equilibrium conditions.

Analysis of deformable bodies

Step 1) Study of forces and equilibrium requirements

→ what different forces are acting

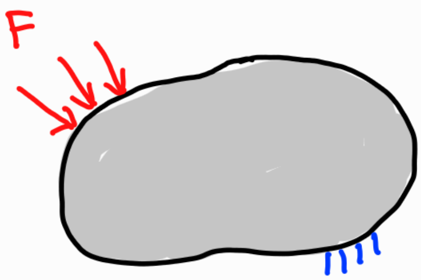
→ draw free-body-diagrams

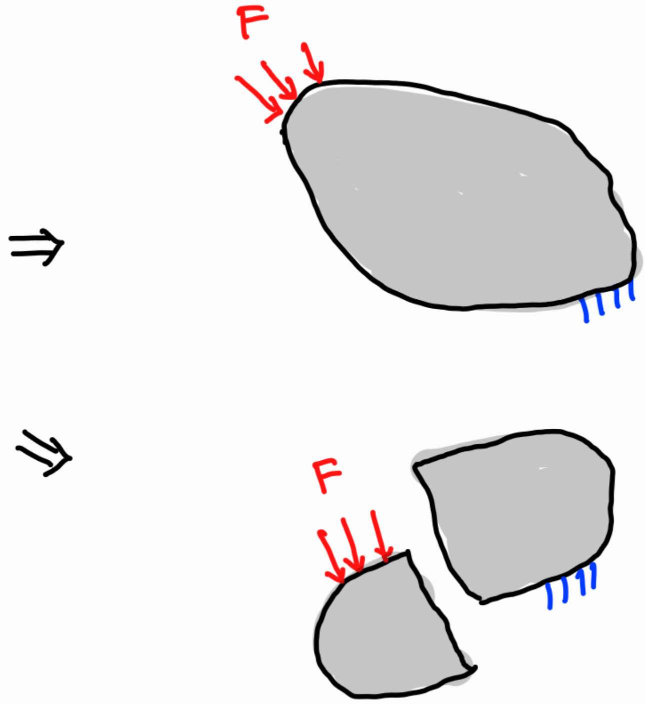
→ satisfy equilibrium conditions

$$\sum F_x = \sum F_y = \sum F_z = 0$$
$$\sum M_x = \sum M_y = \sum M_z = 0$$

→ find reactions

Step 2) Study of deformation and conditions of geometric compatibility





Deformations cannot be arbitrary

must be compatible with the whole system

Incompatible deformations

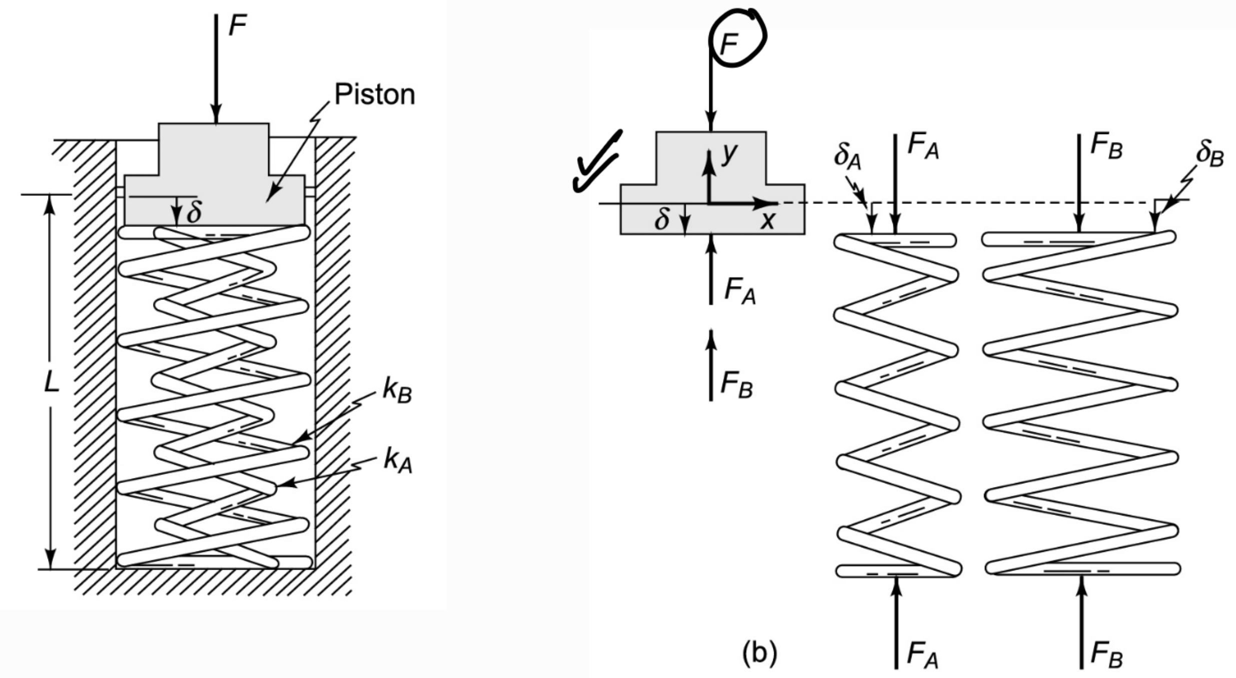
Step 3) Application of force - deformation relations

→ Forces/moments are the cause

→ Deformations are the effect

→ The 3rd step is the study of how the cause and effect are related

Let's take an example to understand these three steps:



1) Study equilibrium

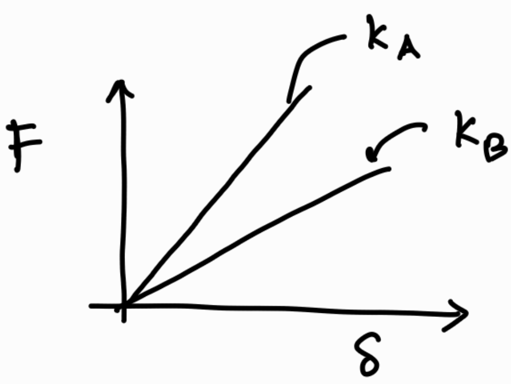
$$+\uparrow \sum F_y = 0 \quad \Rightarrow \quad -F + F_A + F_B = 0$$

$$\Rightarrow \quad F = F_A + F_B$$

2) Geometric compatibility $\quad \delta_A = \delta_B = \delta$ (say)

3) Force-deformation relationship

$\delta_A \sim F_A \qquad\qquad \delta_B \sim F_B$



✓ $F_A = k_A \delta_A$

✓ $F_B = k_B \delta_B$

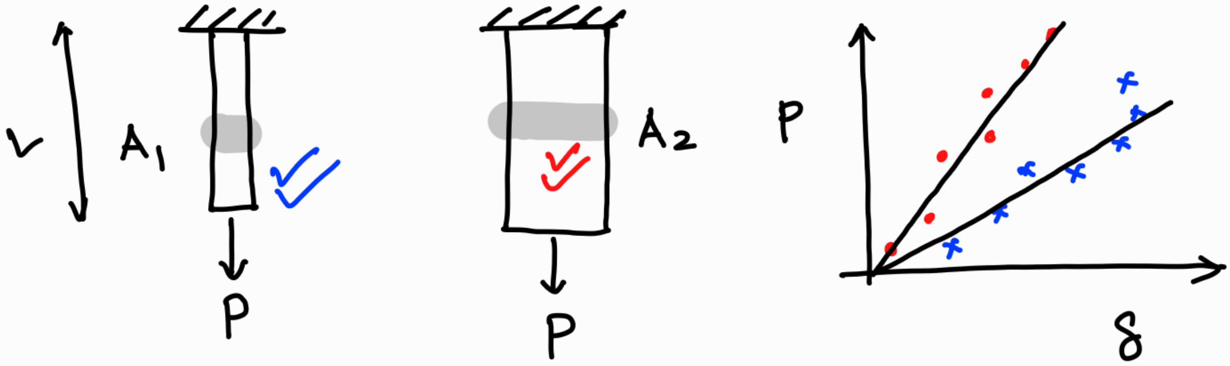
$$F = F_A + F_B = (k_A + k_B)\delta$$

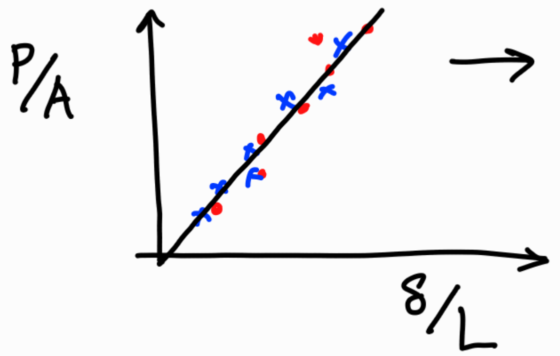
$$\delta_A = \delta_B = \delta$$

$$\frac{F_A}{F} = \frac{k_A \cancel{\delta_A}^{\,\delta}}{(k_A + k_B)\,\delta} = \frac{k_A}{(k_A + k_B)}$$

$$F_A = k_A/(k_A+k_B)\,F \quad , \quad F_B = k_B/(k_A+k_B)\,F$$

Uniaxial loading & deformation





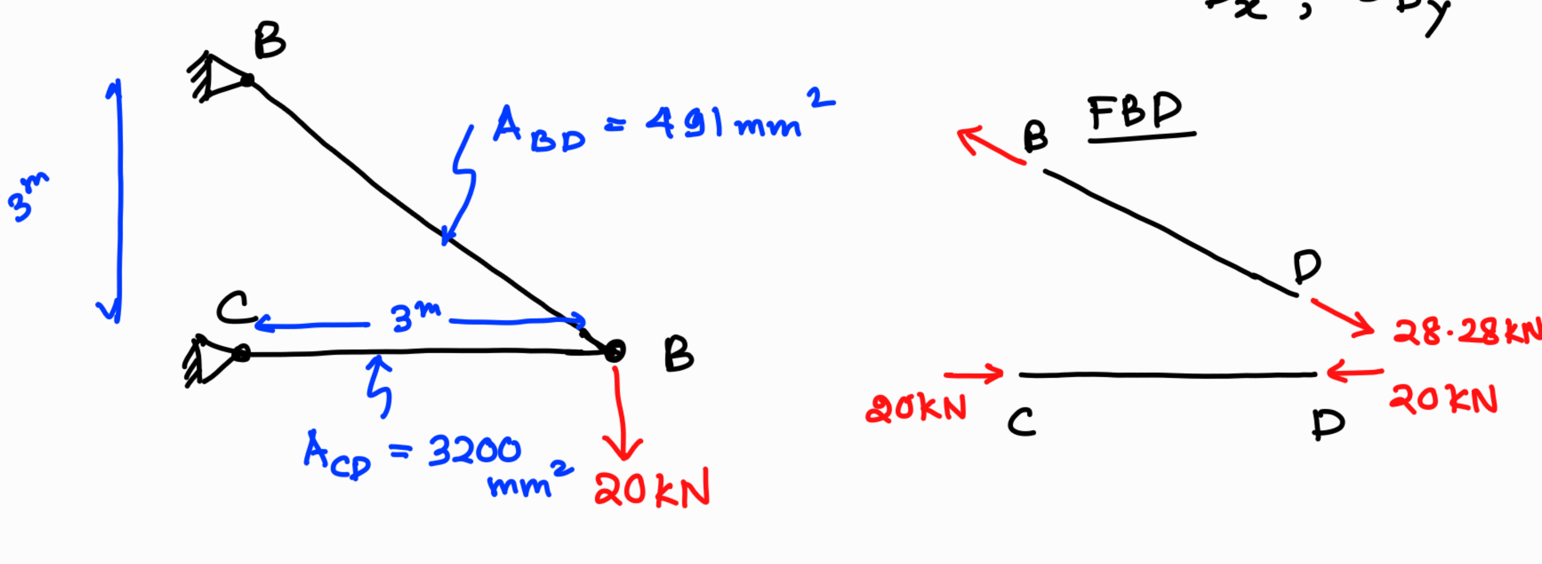
→ Stress-strain relation

$$\frac{P}{A} = E\,\frac{\delta}{L}$$

$$\delta = \frac{PL}{AE}$$

Steel , $E = 205\ kN/mm^2$

Find δ_{Dx} , δ_{Dy}

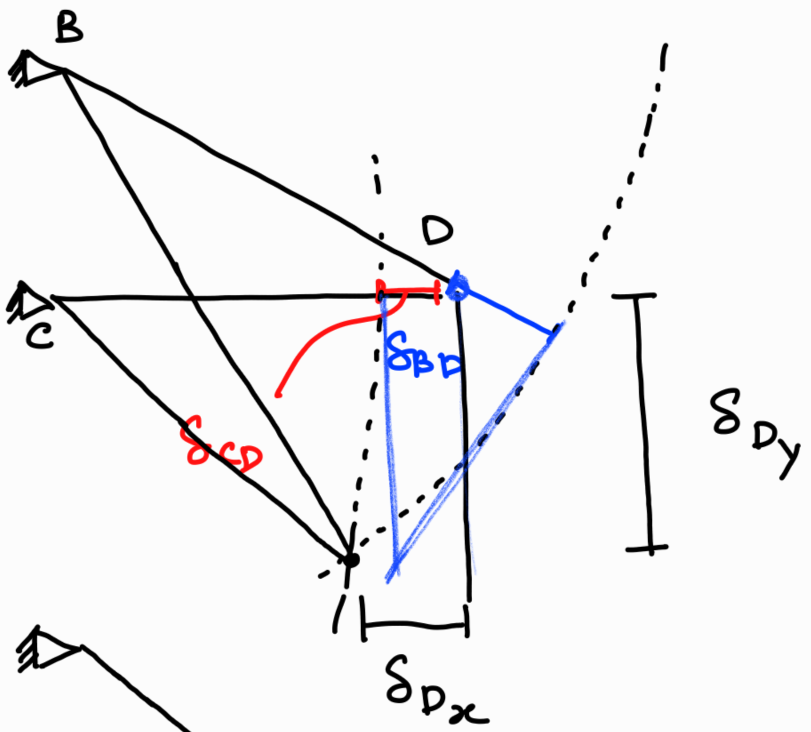


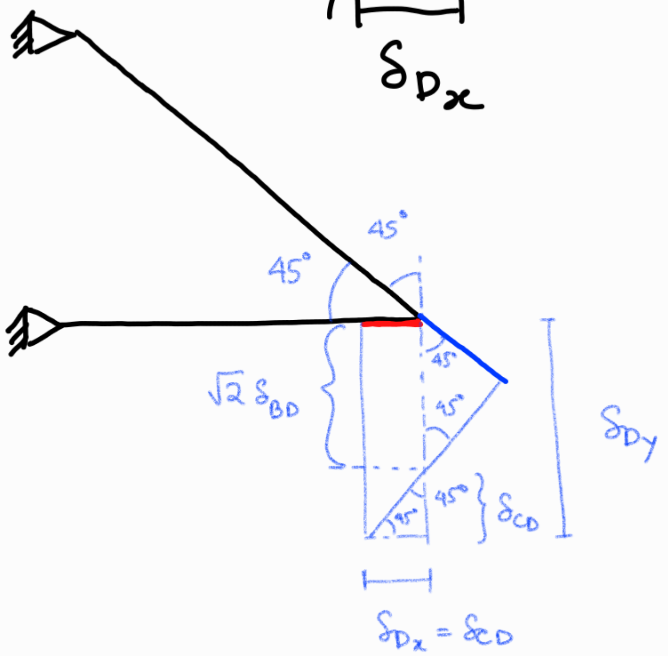
2) Force-deformation (Stress-strain relation)

$$\delta_{BD} = \frac{F_{BD} L_{BD}}{A_{BD} E_{BD}} = 1.19 \text{ mm (Elongation)}$$

$$\delta_{CD} = \frac{F_{CD} L_{CD}}{A_{CD} E_{CD}} = 0.0915 \text{ mm (Compression)}$$

c) Geometric compatibility





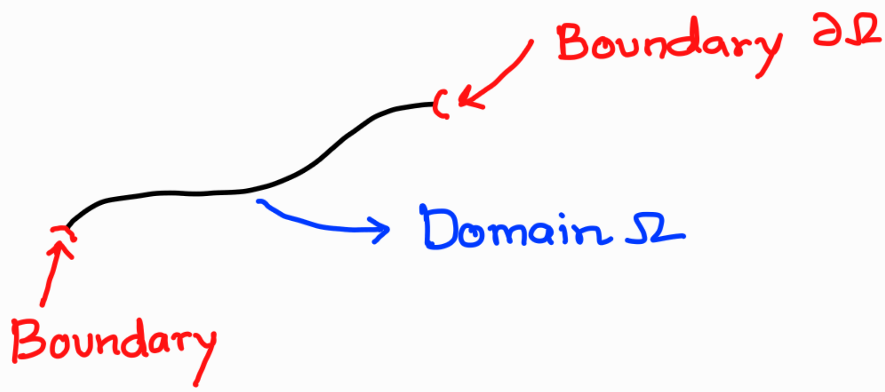
Some terminologies in mechanics

Domain and Boundary

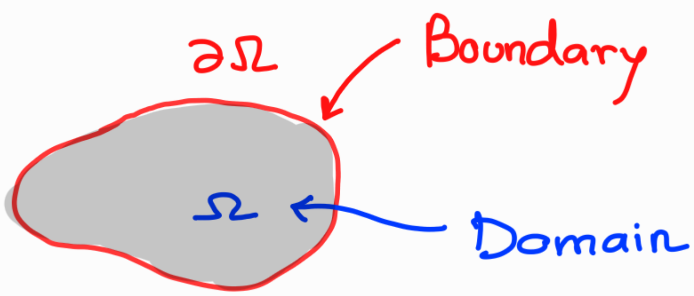
Domain → Region of space occupied by a body

Boundary → Surface/edges that enclose the domain

a) 1D body



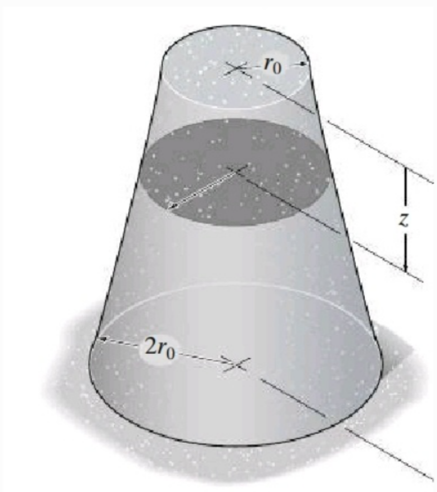
b) 2D body



c) 3D body

Every material point inside the volume is the domain

All the enclosing surfaces form the boundary

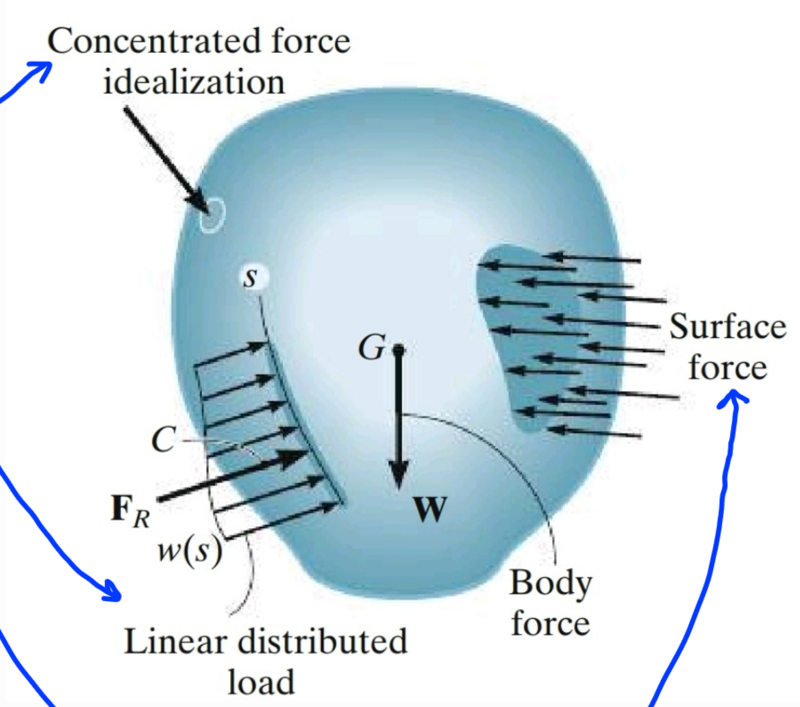


Types of forces

1) EXTERNAL FORCES

a) Surface forces

(contact need with the boundary to apply these forces)



b) Body forces

(forces that act on the body without direct contact)

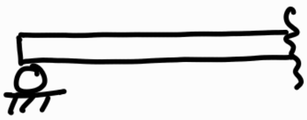
e.g. gravitational force

electromagnetic force

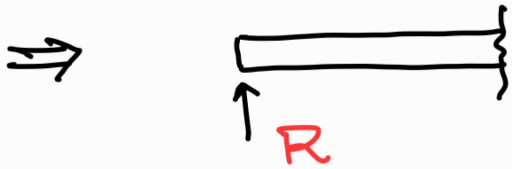
Usually they act on each particle of the body

2) SUPPORT REACTIONS : Surface forces that develop at supports / points of connections

Connection | **Reaction**

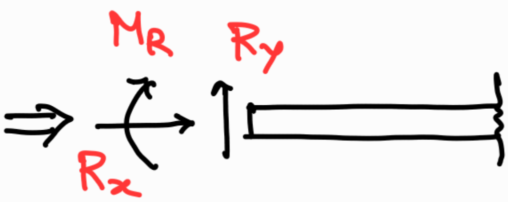


Roller



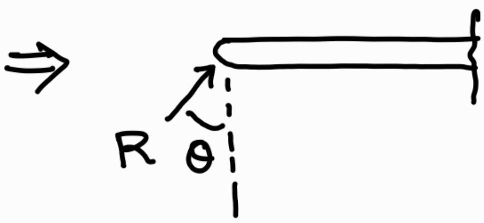
One unknown

Fixed

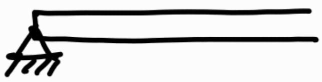


Three unknowns

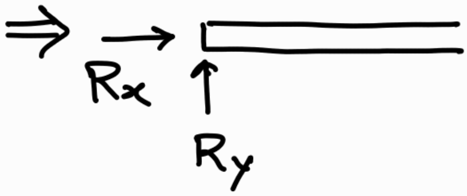
Smooth support



One unknown



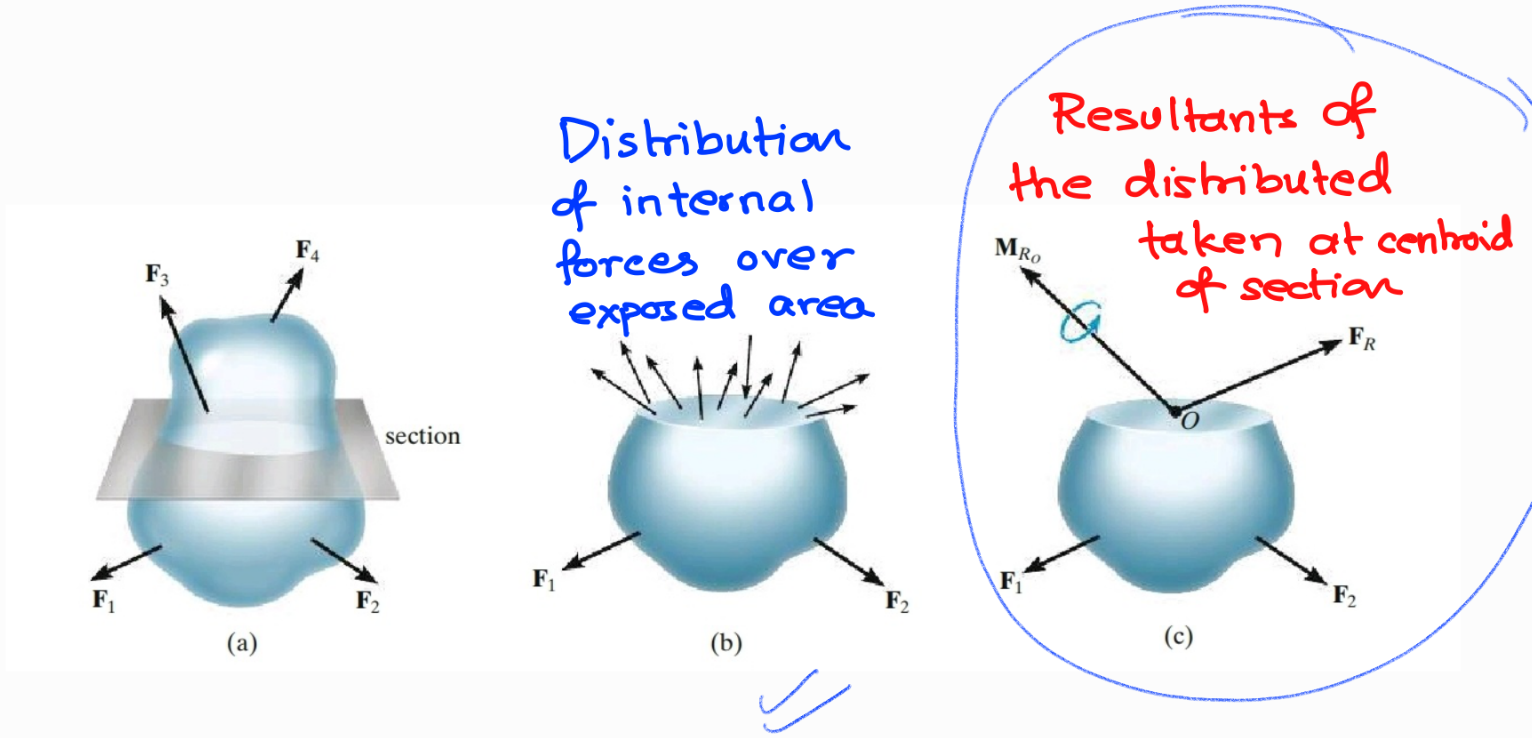
Hinged

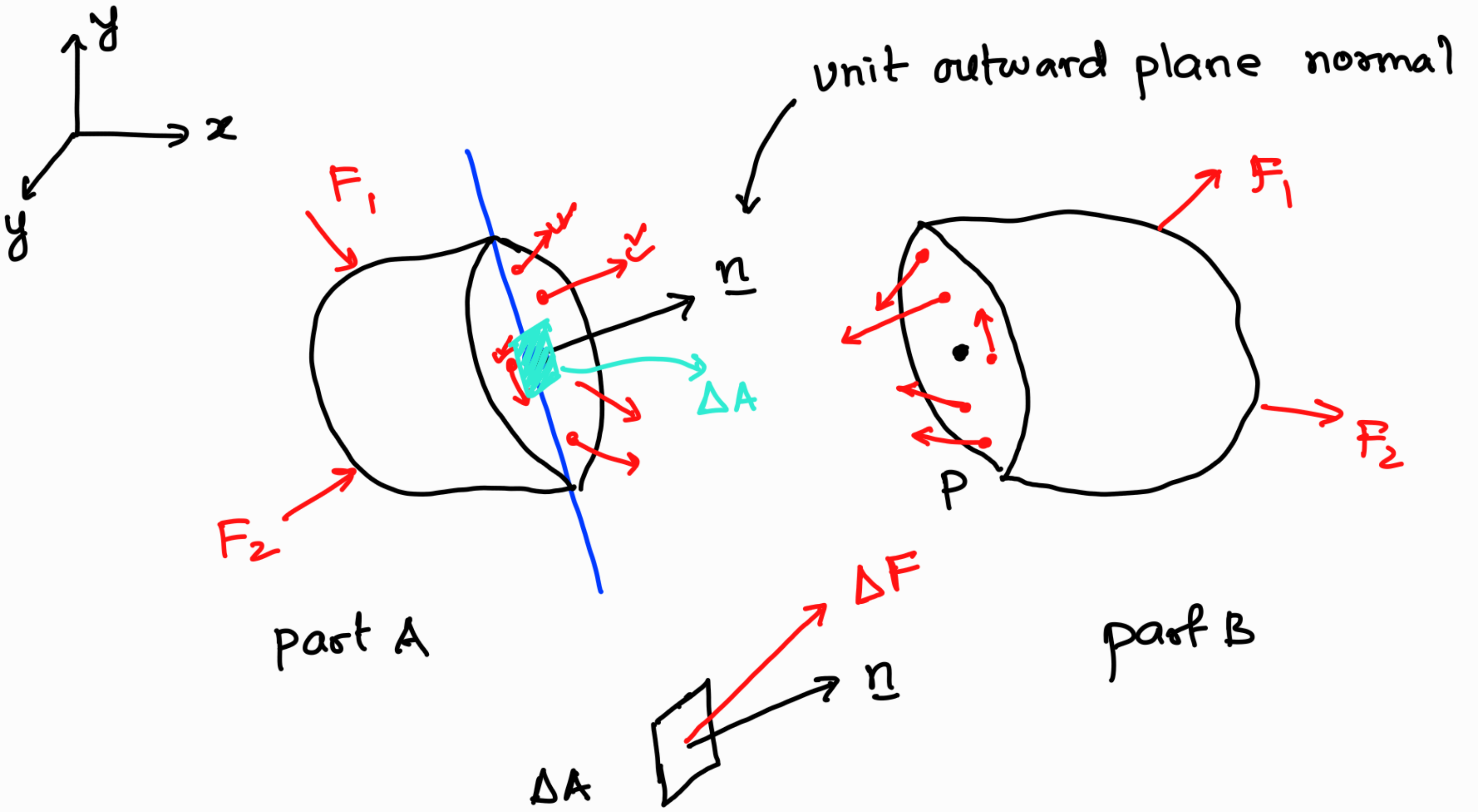


Two unknowns

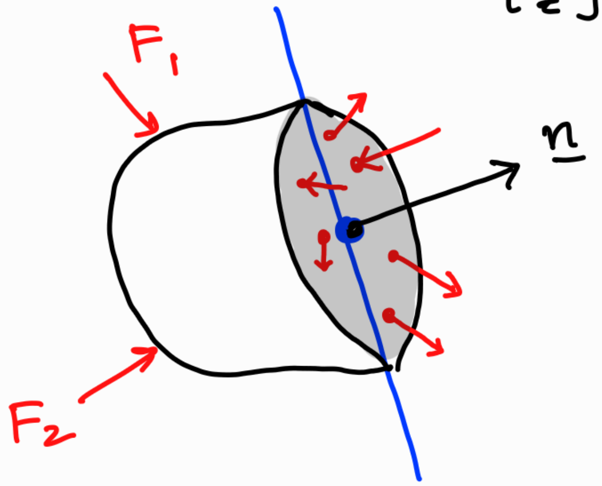
3) INTERNAL RESISTIVE FORCES

These are surface forces that are developed inside a body in resistance to externally applied forces





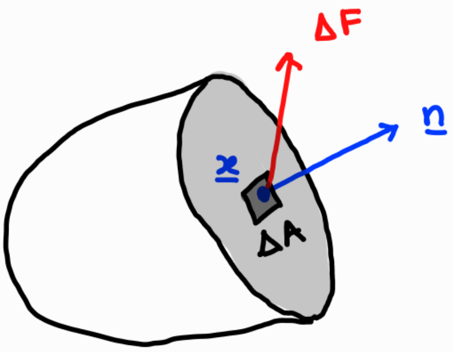
$$\underline{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad T(\underline{x}, \underline{n})$$

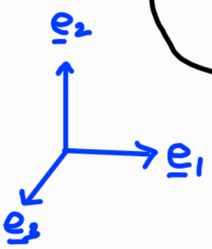


part A

$$T^n(\underline{x}) = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A}$$

If we shrink the area ΔA s.t. the area ΔA always contains the point $\underline{x}$, then the force acting at the point $\underline{x}$ in the limiting case of $\Delta A \to 0$ is called the traction vector (or stress vector)





$\underline{n}$ denotes the outward unit normal of the plane

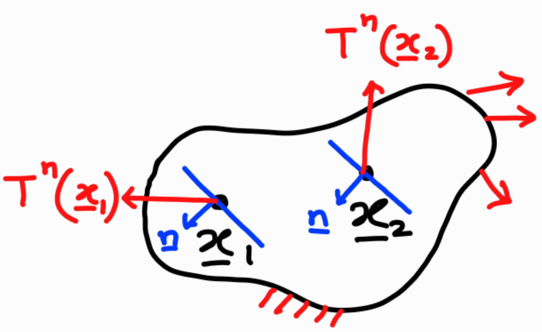
$T^n(\underline{x})$

(or $T(\underline{x}; \underline{n})$)

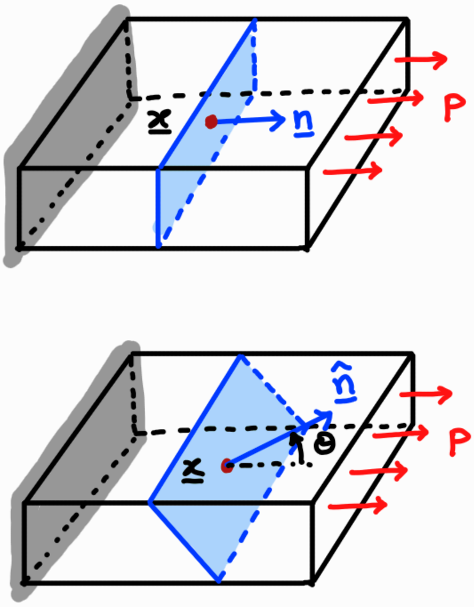
$$\underline{x} = x_1 \underline{e}_1 + x_2 \underline{e}_2 + x_3 \underline{e}_3$$

Remarks

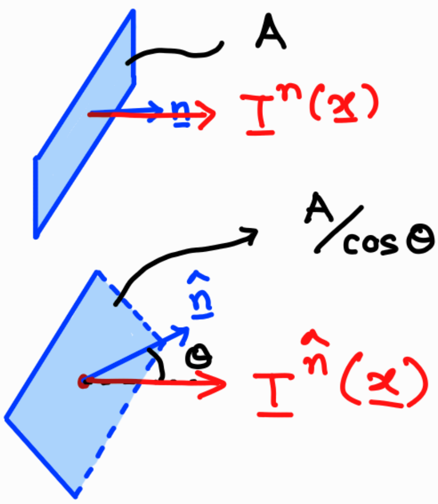
- Traction vector changes from point to point in the body



- Traction vector depends upon the plane orientation



Area



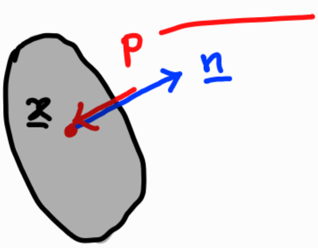
Traction

$$\underline{T}^{n}(\underline{x}) = \frac{P}{A}$$

$$\underline{T}^{\hat{n}}(\underline{x}) = \frac{P}{(A/\cos\theta)} = \frac{P}{A}\cos\theta$$

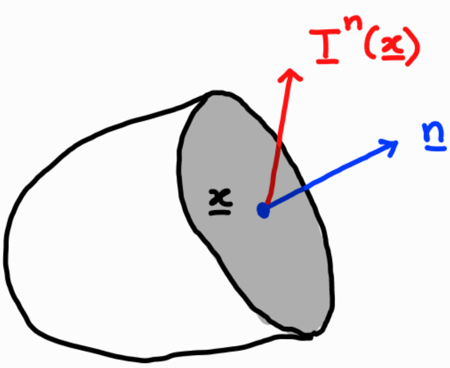
$$\underline{T}^{\hat{n}}(\underline{x}) = \underline{T}^{n}(\underline{x})\,(\underline{n}\cdot\hat{\underline{n}})$$

- Traction vector has same units as pressure (N/m^2) but it is more general than pressure (why?)



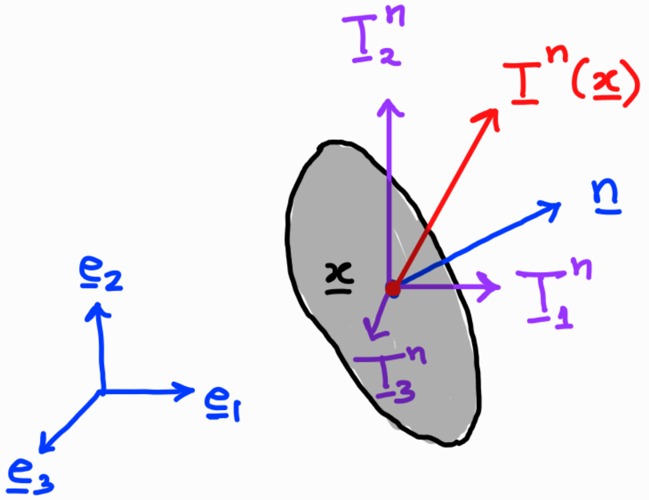
pressure alway acts in the direction opposite to the outward plane normal $\underline{n}$ whereas traction vector can act in any arbitrary direction

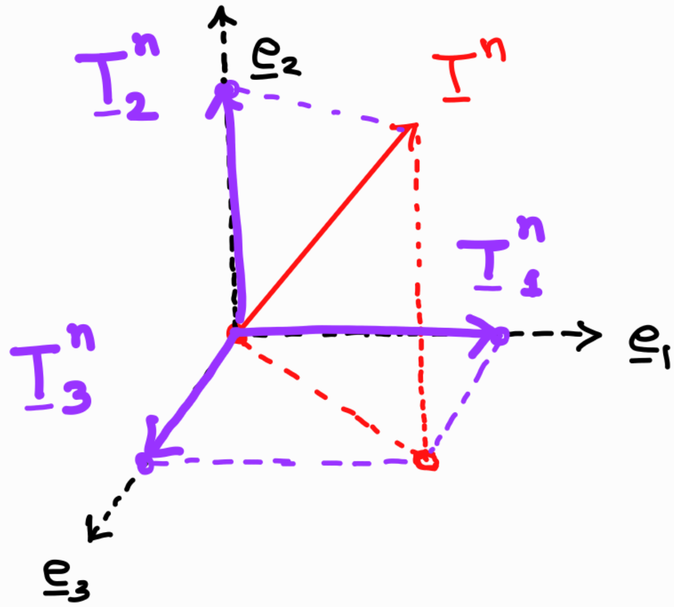
- Traction vector at a point on a plane can have arbitrary direction.



Since traction is a vector, one can obtain components of the vector using a choice of coordinate system

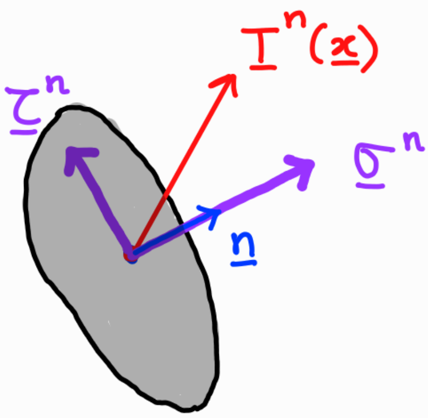
$$\underline{T}^n(\underline{x}) = T_1^n(\underline{x})\,\underline{e}_1 + T_2^n(\underline{x})\,\underline{e}_2 + T_3^n(\underline{x})\,\underline{e}_3$$





$$|\underline{T}^n|^2 = |\underline{T}_1^n|^2 + |\underline{T}_2^n|^2 + |\underline{T}_3^n|^2$$

Normal and shear components of traction vector



Traction vector

$\underline{T}^n(\underline{x})$

resolved along $\underline{n}$ → $\underline{\sigma}^n$

resolved ⊥ to $\underline{n}$ → $\underline{\tau}^n$

$$\underline{\sigma}^n = \underline{T}^n \cdot \underline{n}$$

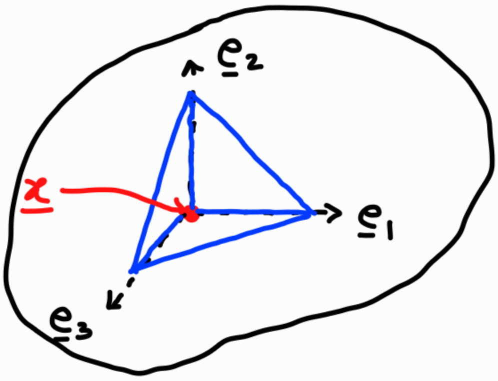
$$\underline{\tau}^n = \underline{T}^n - \underline{\sigma}^n$$

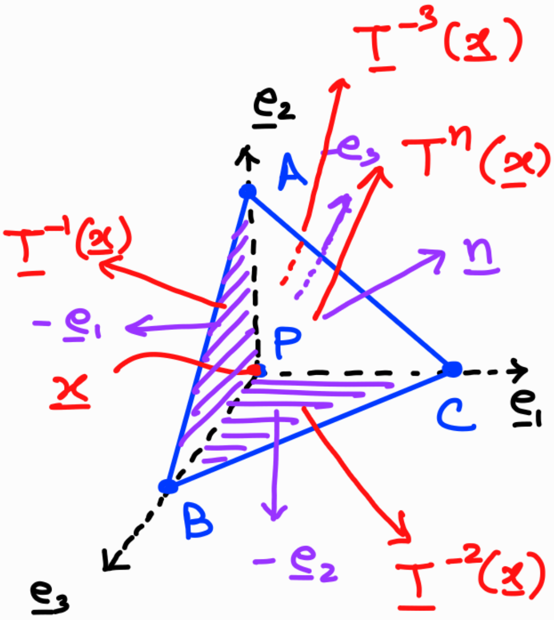
$$|\underline{T}^n|^2 = |\underline{\sigma}^n|^2 + |\underline{\tau}^n|^2$$

Relation of traction on different planes at a pt

We will now prove that if we know traction vectors on three mutually perpendicular planes at a point, we can find traction on any plane at that point.

Imagine a small volume in the shape of a tetrahedron with its vertex at point $\underline{x}$





The tetrahedron has four faces:

Faces		Outward normals
ABC	—	$\underline{n}$
PAB	—	$-\underline{e}_1$
PBC	—	$-\underline{e}_2$
PAC	—	$-\underline{e}_3$

What are the forces acting on the tetrahedron?

— Weight of the tetrahedron (body force) $\rightarrow$ acts through the COM of the tetrahedron

– Internal (surface) forces on the faces of the tetrahedron

$$\sum \underline{F} = 0$$

$$\Rightarrow \underline{T}^{-1} A_{PAB} + \underline{T}^{-2} A_{PBC} + \underline{T}^{-3} A_{PAC} + \underline{T}^{n} A_{ABC} + \rho V \underline{g} = 0$$

Express A_{PAB}, A_{PBC}, A_{PAC} in terms of A_{ABC}

$$A_{PAB} = A_{ABC}\,(\underline{n}\cdot\underline{e}_1)$$

$$A_{PBC} = A_{ABC}\,(\underline{n}\cdot\underline{e}_2)$$

$$A_{PAC} = A_{ABC}\,(\underline{n}\cdot\underline{e}_3)$$

$$\underline{T}^{-1} \cancel{A_{ABC}}\,(\underline{n}\cdot\underline{e}_1) + \underline{T}^{-2} \cancel{A_{ABC}}\,(\underline{n}\cdot\underline{e}_2) + \underline{T}^{-3} \cancel{A_{ABC}}\,(\underline{n}\cdot\underline{e}_3) + \underline{T}^{n} \cancel{A_{ABC}} + \frac{1}{3}\rho \underline{g}\,(\cancel{A_{ABC}}\cdot h) = \underline{0}$$

$$\Rightarrow \underline{T}^{-1}(\underline{n}\cdot\underline{e}_1) + \underline{T}^{-2}(\underline{n}\cdot\underline{e}_2) + \underline{T}^{-3}(\underline{n}\cdot\underline{e}_3) + \underline{T}^{n} + \frac{\rho \underline{g}\, h}{3} = \underline{0}$$

$$\Rightarrow \sum_{i=1}^{3} \underline{T}^{-i}\,(\underline{n}\cdot\underline{e}_i) + \underline{T}^{n} + \frac{1}{3}\rho \underline{g} h = \underline{0}$$

Now our goal is to find tractions at the pt P, instead of the faces. So we reduce 'h' to zero s.t. the tetrahedron shrinks to the point P

As $h \to 0$, the term $\rho g h \to \underline{0}$ and we have a simple result:

$$\sum_{i=1}^{3} \underline{T}^{-i} (\underline{n} \cdot \underline{e}_i) + \underline{T}^{n} = \underline{0}$$

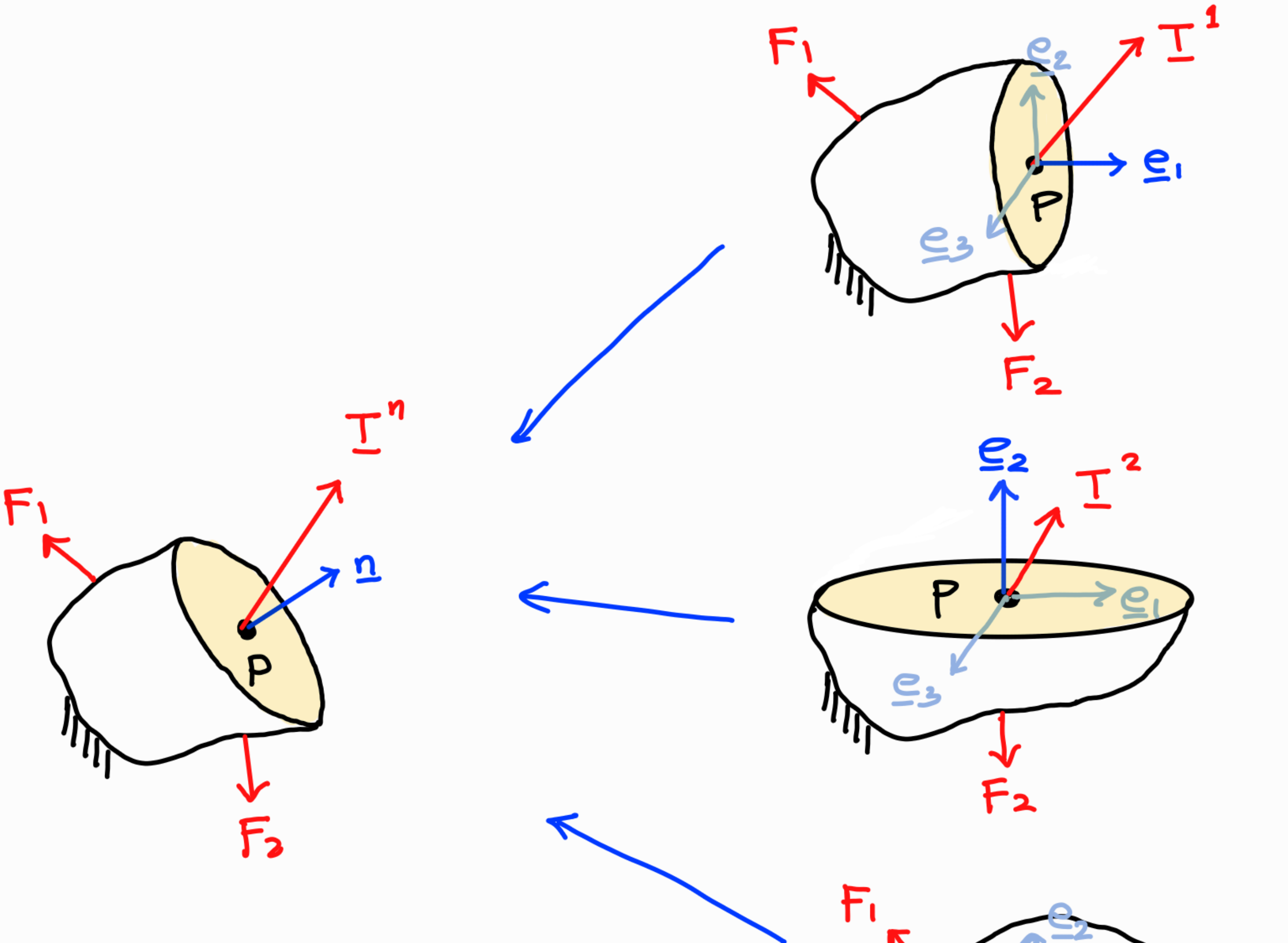
We already know that $\boxed{\underline{T}^{-i} = -\underline{T}^{i}}$

$$\underline{T}^{n}(\underline{x}) = \sum_{i=1}^{3} \underline{T}^{i} (\underline{n} \cdot \underline{e}_i)$$

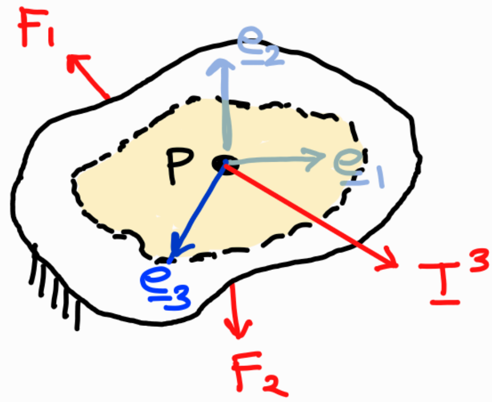
depends on your choice of coordinate system

If we know the traction vectors on three mutually perpendicular planes, then the traction vector on any plane passing through the point P can be obtained using the above relation

The body force term dropped out from the above formula — no approximation was made! Thus, this formula holds even if body force is present



$$\underline{T}^n = \underline{T}^1(\underline{n}\cdot\underline{e}_1) + \underline{T}^2(\underline{n}\cdot\underline{e}_2) + \underline{T}^3(\underline{n}\cdot\underline{e}_3)$$

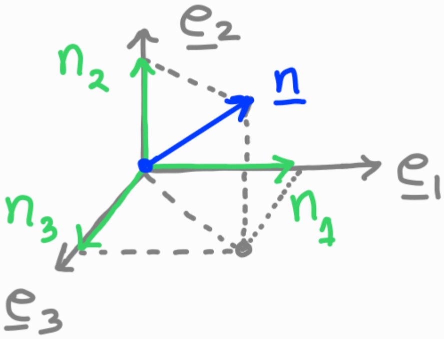


For pt $\underline{x}$,

$$\underline{T}^n = \underline{T}^1(\underline{n}\cdot\underline{e}_1) + \underline{T}^2(\underline{n}\cdot\underline{e}_2) + \underline{T}^3(\underline{n}\cdot\underline{e}_3)$$

$$= \underline{T}^1 n_1 + \underline{T}^2 n_2 + \underline{T}^3 n_3$$

direction cosines
(not vectors)

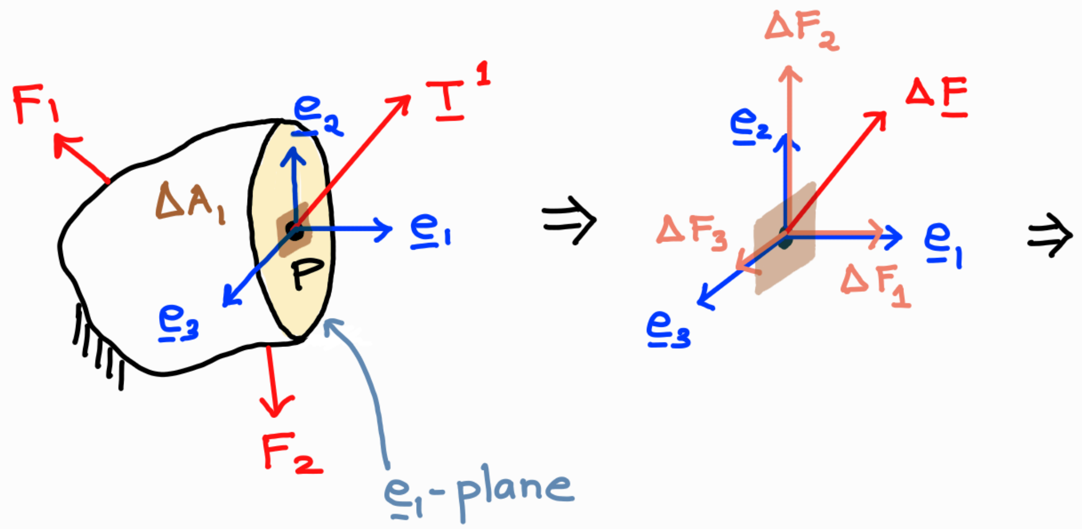


What do the tractions $\underline{T}^1, \underline{T}^2, \underline{T}^3$ represent?

$\underline{T}^1$ acts on a plane with outward normal $\underline{e}_1$

Lets cut a $\underline{e}_1$-plane through point P

(plane with outward normal $\underline{e}_1$)



$$T_1^1 = \lim_{\Delta A_1 \to 0} \frac{\Delta F_1}{\Delta A_1}$$

$$T_2^1 = \lim_{\Delta A_1 \to 0} \frac{\Delta F_2}{\Delta A_1}$$

$$T_3^1 = \lim_{\Delta A_1 \to 0} \frac{\Delta F_3}{\Delta A_1}$$

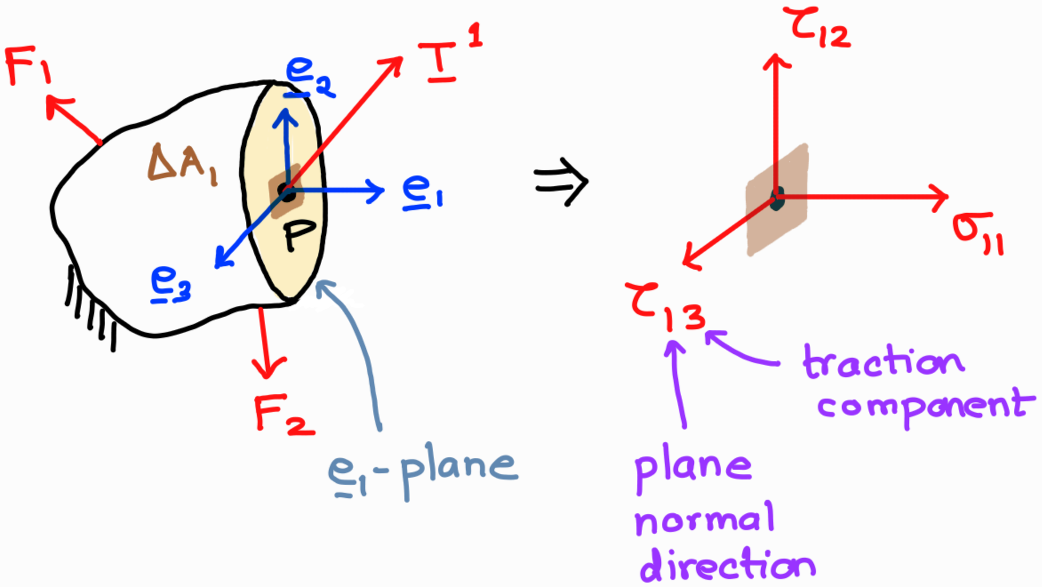
$$\underline{T}^1 = T_1^1 \underline{e}_1 + T_2^1 \underline{e}_2 + T_3^1 \underline{e}_3$$

If we use the normal & shear components decomposition of $\underline{T}^1$, we can see that:

Normal stress component, $\sigma_{11} = T_1^1$ } tendency to pull or push

Shear stress components, $\tau_{12} = T_2^1$, $\tau_{13} = T_3^1$ } tendency to slide between two surfaces

Similarly, we can define for $\underline{T}^2$ and $\underline{T}^3$



$\tau_{ij} \rightarrow$ represents the jth component of traction on the ith plane

So the traction vector $\underline{T}^1$ can be represented in the reference frame $\underline{e}_1 - \underline{e}_2 - \underline{e}_3$ as

$$[\underline{T}^1]_{\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}} = \begin{bmatrix} T_1^1 \\ T_2^1 \\ T_3^1 \end{bmatrix} = \begin{bmatrix} \sigma_{11} \\ \tau_{12} \\ \tau_{13} \end{bmatrix}$$

or

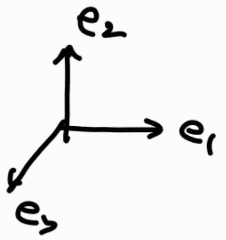
$$\underline{T}^1 = \sigma_{11}\, \underline{e}_1 + \tau_{12}\, \underline{e}_2 + \tau_{13}\, \underline{e}_3$$

$$\underline{T}^2 = \tau_{21}\, \underline{e}_1 + \sigma_{22}\, \underline{e}_2 + \tau_{23}\, \underline{e}_3$$

$$\underline{T}^3 = \tau_{31}\, \underline{e}_1 + \tau_{32}\, \underline{e}_2 + \sigma_{33}\, \underline{e}_3$$

$$\underline{\underline{\underline{T}^n = \underline{T}^1 n_1 + \underline{T}^2 n_2 + \underline{T}^3 n_3}}}$$

$$\left(\underline{T}^n\right)_{\begin{pmatrix}e_1\\e_2\\e_3\end{pmatrix}} = \begin{bmatrix} T_1^n \\ T_2^n \\ T_3^n \end{bmatrix} \quad , \quad \left(\underline{T}^1\right)_{\begin{pmatrix}e_1\\e_2\\e_3\end{pmatrix}} = \begin{bmatrix} \sigma_{11} \\ \tau_{12} \\ \tau_{13} \end{bmatrix} \quad , \quad \dots \quad \left(\underline{T}^3\right) = \begin{bmatrix} \tau_{31} \\ \tau_{32} \\ \sigma_{33} \end{bmatrix}$$



$$\underbrace{\begin{bmatrix} T_1^n \\ T_2^n \\ T_3^n \end{bmatrix}}_{\underline{T}^n} = \underbrace{\begin{bmatrix} \sigma_{11} \\ \tau_{12} \\ \tau_{13} \end{bmatrix}}_{\underline{T}^1} n_1 + \underbrace{\begin{bmatrix} \tau_{21} \\ \sigma_{22} \\ \tau_{23} \end{bmatrix}}_{\underline{T}^2} n_2 + \underbrace{\begin{bmatrix} \tau_{31} \\ \tau_{32} \\ \sigma_{33} \end{bmatrix}}_{\underline{T}^3} n_3$$

$$\underbrace{\begin{bmatrix} T_1^n \\ T_2^n \\ T_3^n \end{bmatrix}}_{\left(\underline{T}^n\right)_{\begin{pmatrix}e_1\\e_2\\e_3\end{pmatrix}}} = \underbrace{\begin{bmatrix} \sigma_{11} & \tau_{21} & \tau_{31} \\ \tau_{12} & \sigma_{22} & \tau_{32} \\ \tau_{13} & \tau_{23} & \sigma_{33} \end{bmatrix}}_{\left(\sigma\right)_{\begin{pmatrix}e_1\\e_2\\e_3\end{pmatrix}}} \underbrace{\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}}_{\left(\underline{n}\right)_{\begin{pmatrix}e_1\\e_2\\e_3\end{pmatrix}}}$$

$$\boxed{\underline{T}^n = \underline{\underline{\sigma}} \cdot \underline{n}}$$

Stress tensor only depends upon $\underline{x}$ but is independent of the plane normal