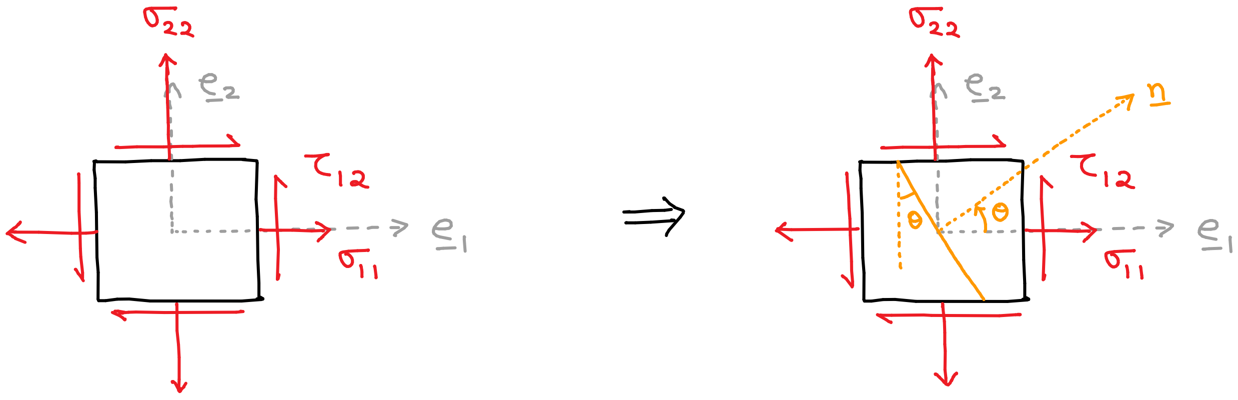
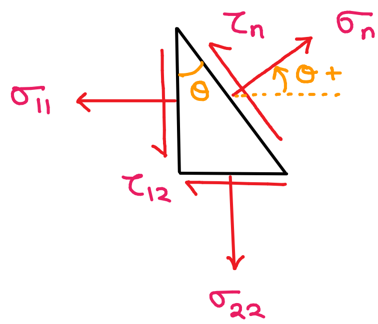
Plane Stress Transformation

$\theta \rightarrow +ve$ (counter-clockwise)



The normal stress on the $\underline{n}$-plane is



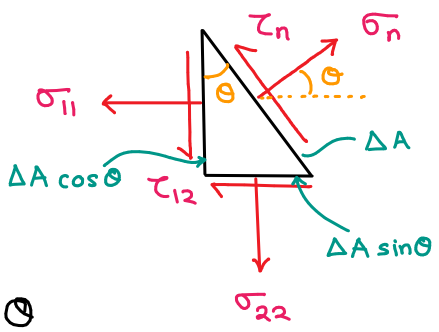
$$\sigma_n = \sigma_{11}\, n_1^2 + \sigma_{22}\, n_2^2 + 2\tau_{12}\, n_1 n_2$$

$$= \sigma_{11} \cos^2\theta + \sigma_{22} \sin^2\theta + 2\tau_{12} \cos\theta \sin\theta$$

$$= \sigma_{11} \frac{(1+\cos 2\theta)}{2} + \sigma_{22} \frac{(1-\cos 2\theta)}{2} + \tau_{12} \sin 2\theta$$

$$= \frac{\sigma_{11}+\sigma_{22}}{2} + \frac{\sigma_{11}-\sigma_{22}}{2} \cos 2\theta + \tau_{12} \sin 2\theta$$

The above form can also be arrived at by doing force equilibrium of the wedge in the $\underline{n}$-direction



$$\nwarrow^{+} \sum F_n = 0$$

$$\Rightarrow \quad \sigma_n \Delta A - (\tau_{12} \Delta A \sin\theta) \cos\theta$$

$$- (\sigma_{22} \Delta A \sin\theta) \sin\theta - (\tau_{12} \Delta A \cos\theta) \sin\theta$$

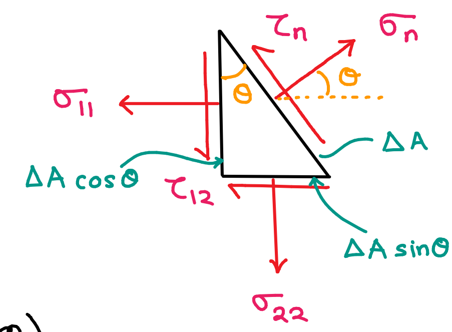
$$- (\sigma_{11} \Delta A \cos\theta) \cos\theta = 0$$

$$\Rightarrow \quad \sigma_n = \sigma_{11} \cos^2\theta + \sigma_{22} \sin^2\theta + 2\tau_{12} \sin\theta \cos\theta$$

$$= \frac{\sigma_{11}+\sigma_{22}}{2} + \frac{\sigma_{11}-\sigma_{22}}{2} \cos 2\theta + \tau_{12} \sin 2\theta$$

The shear stress on the $\underline{n}$-plane can be obtained from the geometry conveniently by doing force equilibrium of the wedge in the direction perpendicular to $\underline{n}$

$$+\nwarrow \sum F_{n\perp} = 0$$



$$\Rightarrow \tau_n \Delta A + \left(\tau_{12}\, \Delta A \sin\theta\right) \sin\theta - \left(\sigma_{22}\, \Delta A \sin\theta\right) \cos\theta$$

$$+ \left(\sigma_{11}\, \Delta A \cos\theta\right) \sin\theta - \left(\tau_{12}\, \Delta A \cos\theta\right) \cos\theta = 0$$

$$\Rightarrow \tau_n = \left(\sigma_{22} - \sigma_{11}\right) \sin\theta \cos\theta + \tau_{12} \left(\cos^2\theta - \sin^2\theta\right)$$

$$\Rightarrow \tau_n = -\frac{\left(\sigma_{11} - \sigma_{22}\right)}{2} \sin 2\theta + \tau_{12} \cos 2\theta$$

So we get two relations for σ_n and τ_n in plane stress

$$\sigma_n = \frac{\sigma_{11} + \sigma_{22}}{2} + \frac{\sigma_{11} - \sigma_{22}}{2} \cos 2\theta + \tau_{12} \sin 2\theta$$

$$\tau_n = -\frac{\sigma_{11} - \sigma_{22}}{2} \sin 2\theta + \tau_{12} \cos 2\theta$$

Maximum / Minimum normal stress planes

To determine the orientation that causes the normal stress to be maximum / minimum, we can take a derivative of σ_n w.r.t to plane inclination θ

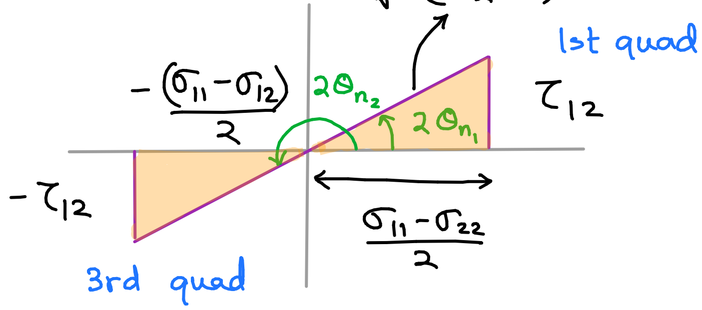
$$\frac{d\sigma_n}{d\theta} = -\frac{\sigma_{11} - \sigma_{22}}{2} \left(2 \sin 2\theta\right) + 2\tau_{12} \cos 2\theta = 0$$

Solving the above equation, we obtain the orientation $\theta = \theta_n$ of the planes of maximum and minimum normal stress

Assume $\sigma_{11} > \sigma_{22}$, $\tan \phi \to +ve \begin{cases} \text{1st quad} \\ \text{3rd quad} \end{cases}$

$$R = \sqrt{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \tau_{12}^2}$$

$$\tan 2\theta_n = \frac{\tau_{12}}{\left(\frac{\sigma_{11} - \sigma_{12}}{2}\right)}$$



The solution has two roots θ_{n_1} and θ_{n_2}. Specifically, the values of $2\theta_{n_1}$ and $2\theta_{n_2}$ are 180° apart, so in reality θ_{n_1} and θ_{n_2} planes will be 90° apart.

To obtain maximum and minimum normal stress, we must substitute the angles θ_{n_1} and θ_{n_2}.

$$\sigma_{n_1, n_2} = \frac{\sigma_{11} + \sigma_{22}}{2} \pm \frac{\sigma_{11} - \sigma_{22}}{2} \cos 2\theta_{n_1} \pm \tau_{12} \sin 2\theta_{n_1}$$

$$= \frac{\sigma_{11} + \sigma_{22}}{2} \pm \frac{\sigma_{11} - \sigma_{22}}{2} \left(\frac{\sigma_{11} - \sigma_{22}}{2R}\right) \pm \tau_{12} \left(\frac{\tau_{12}}{R}\right)$$

$$= \frac{\sigma_{11} + \sigma_{22}}{2} \pm \frac{(\sigma_{11} - \sigma_{22})^2}{4R} \pm \frac{\tau_{12}^2}{R}$$

$$= \frac{\sigma_{11} + \sigma_{22}}{2} \pm \frac{1}{R} \left[\underbrace{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \tau_{12}^2}_{R^2} \right]$$

$$= \frac{\sigma_{11} + \sigma_{22}}{2} \pm \sqrt{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \tau_{12}^2}$$

This result gives the maximum and minimum normal s ess acting at a point and the angles $\theta_{n_1}, \theta_{n_2}$ are the directions

$$\sigma_{max}\Big|_{\text{plane normal } \underline{n}_1} = \frac{\sigma_{11}+\sigma_{22}}{2} + \sqrt{\left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)^2 + \tau_{12}^2}$$

$$\sigma_{min}\Big|_{\text{plane normal } \underline{n}_2} = \frac{\sigma_{11}+\sigma_{22}}{2} - \sqrt{\left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)^2 + \tau_{12}^2}$$

Lets us also check what are the shear stresses on planes where normal stress components are maximum or minimum

For this, we can put value of $\sin 2\theta_p$ and $\cos 2\theta_p$ in the relation for shear stress component τ_n :

$$\tau_n = -\frac{\sigma_{11}-\sigma_{22}}{2}\sin 2\theta_p + \tau_{12}\cos 2\theta_p$$

$$= -\frac{\sigma_{11}-\sigma_{22}}{2}\left(\frac{\tau_{12}}{R}\right) + \tau_{12}\left(\frac{\sigma_{11}-\sigma_{22}}{2R}\right)$$

$$= 0$$

We find that the shear stress components are zero on planes where normal stress components are maximized or minimized. Coincidentally, principal planes are also planes where there are only normal stress components and no shear stresses. Thus, principal planes are also planes where the normal stresses are max/min and shear stresses are zero

Maximum Shear Stress and corresponding plane

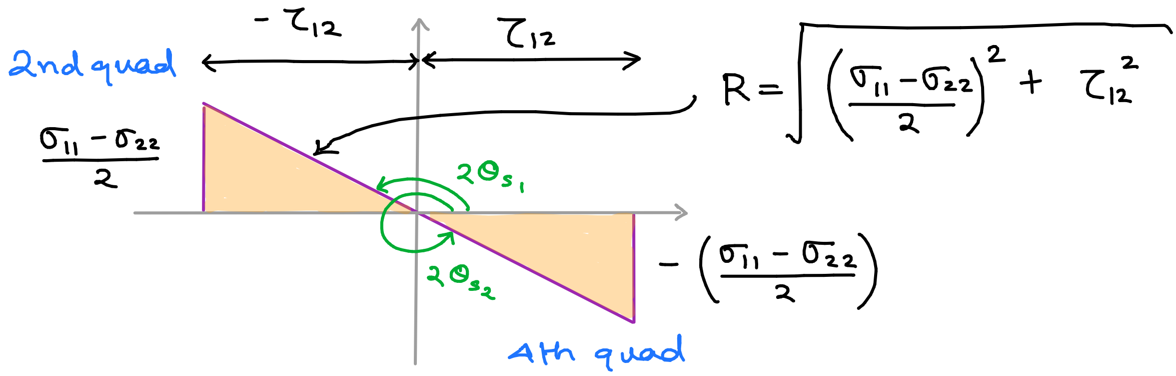
$$\tau_n = -\frac{\sigma_{11} - \sigma_{22}}{2} \sin 2\theta + \tau_{12} \cos 2\theta$$

To find the maximum or minimum value of τ_n w.r.t θ, set

$$\frac{\partial \tau_n}{\partial \theta} = 0 \Rightarrow \boxed{\tan 2\theta_s = -\frac{(\sigma_{11} - \sigma_{12})}{2\tau_{12}}}$$

Without loss of generality, assume $\sigma_{11} > \sigma_{22}$

$\tan \phi$ is -ve in 2nd and 4th quadrant, therefore $2\theta_s$ must belong to either the 2nd or 4th quadrant.



There are two planes $2\theta_{s_1}$ & $2\theta_{s_2}$, where $\tan 2\theta_s$ is -ve

These planes look 180° apart, however, in reality, they are 90° apart.

By comparison, $\tan 2\theta_s$ is the negative reciprocal of $\tan 2\theta_n$, so each plane $2\theta_s$ must be 90° from $2\theta_n$, and in reality the principal planes and the planes of max/min shear occur at angles of 45° to each other

The max/min value of shear stress is obtained by putting the values of θ_s in the relation of τ_n

$$\boxed{\tau_{n,\substack{max\\min}} = \pm\sqrt{\left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)^2 + \tau_{12}^2}}$$

If one uses the coordinate system of principal stresses, one will get max/min shear stresses by $\tau_{12} = 0$, and $\sigma_{11} = \lambda_1$, $\sigma_{22} = \lambda_2$

$$\tau_{max/min} = \pm\left|\frac{\lambda_1 - \lambda_2}{2}\right|$$

What are the normal stresses on the planes of max/min shear stress?

Set the values of θ_{s_1} and θ_{s_2} in the relation for σ_n:

$$\sigma_n = \frac{\sigma_{11}+\sigma_{22}}{2} + \frac{\sigma_{11}-\sigma_{22}}{2}\cos 2\theta_s + \tau_{12}\sin 2\theta_s$$

$$= \frac{\sigma_{11}+\sigma_{22}}{2} + \cancel{\frac{\sigma_{11}-\sigma_{22}}{2}\left(-\frac{\tau_{12}}{R}\right)} + \cancel{\tau_{12}\left(\frac{\sigma_{11}-\sigma_{22}}{2R}\right)}$$

$$= \frac{\sigma_{11}+\sigma_{22}}{2} \quad \text{(average stress)}$$