Principal Stresses and Principal planes

We have seen how to find out normal & shear stress on any plane with normal $\underline{n}$. From failure considerations of materials, it would be of interest to know:

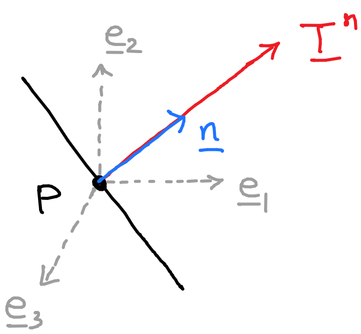
(1) If there are any planes passing through a given point on which the traction vector is wholly normal? (In other words, the traction vector only has non-zero normal component and zero shear components)

(2) On which plane does the normal stress become maximum? What will be the magnitude?

(3) On which plane does the shear stress become maximum? What will be the magnitude?

Lets try to answer these questions.

Consider a plane with normal $\underline{n}$ s.t. the traction vector is oriented along the normal vector.



$$[\underline{T}^n] = \lambda\,[\underline{n}] \quad \text{— (A)}$$

We have also learned that traction on an arbitrary plane can be obtained as

$$[\underline{T}^n] = [\underline{\sigma}]\,[\underline{n}] \quad \text{— (B)}$$

Equating (A) and (B), we get

$$[\underline{\underline{\sigma}}]\,[\underline{n}] = \lambda\,[\underline{n}]$$

eigenvalue

eigenvector

⇐ An eigenvalue problem

$$\Rightarrow \quad [\underline{\underline{\sigma}} - \lambda \underline{\underline{I}}]\,[\underline{n}] = \underline{0}$$

$$\Rightarrow \quad \begin{bmatrix} \sigma_{11}-\lambda & \tau_{12} & \tau_{13} \\ \tau_{12} & \sigma_{22}-\lambda & \tau_{23} \\ \tau_{13} & \tau_{23} & \sigma_{33}-\lambda \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{— (C)}$$

A trivial solution would be $n_1 = n_2 = n_3 = 0$. For the existence of a non-trivial solution, the determinant should be set to zero

$$\begin{vmatrix} \sigma_{11}-\lambda & \tau_{12} & \tau_{13} \\ \tau_{12} & \sigma_{22}-\lambda & \tau_{23} \\ \tau_{13} & \tau_{23} & \sigma_{33}-\lambda \end{vmatrix} = 0$$

Expanding the above determinant, we get

$$\lambda^3 - (\sigma_{11} + \sigma_{22} + \sigma_{33})\,\lambda^2$$
$$+ \left(\sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{11}\sigma_{33} - \tau_{12}^2 - \tau_{13}^2 - \tau_{23}^2\right)\lambda$$
$$- \left(\sigma_{11}\sigma_{22}\sigma_{33} + 2\tau_{12}\tau_{23}\tau_{13} - \sigma_{11}\tau_{23}^2 - \sigma_{22}\tau_{13}^2 - \sigma_{33}\tau_{12}^2\right) = 0$$

There are three roots of the cubic equation

$$\rightarrow \left.\lambda_1, \lambda_2, \lambda_3\right\} \quad \text{3 eigenvalues}$$

Substituting each eigenvalue one by one in (C) would lad to getting the corresponding n_1, n_2, n_3. Also use $n_1^2 + n_2^2 + n_3^2 = 1$

Substitute $\lambda = \lambda_1$ and solve for $\underline{n}_1$ ← eigenvector associated with eigenvalue λ_1

$$\begin{bmatrix} \sigma_{11} - \lambda_1 & \tau_{12} & \tau_{13} \\ \tau_{12} & \sigma_{22} - \lambda_1 & \tau_{23} \\ \tau_{13} & \tau_{23} & \sigma_{33} - \lambda_1 \end{bmatrix} \underbrace{\begin{bmatrix} n_{11} \\ n_{21} \\ n_{31} \end{bmatrix}}_{[\underline{n}_1]} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Terminology:

λ_1 – 1st principal stress , $[\underline{n}_1]$ – 1st principal plane

Substitute $\lambda = \lambda_2$ and solve for $\underline{n}_2$ ← eigenvector associated with eigenvalue λ_2

$$\begin{bmatrix} \sigma_{11} - \lambda_2 & \tau_{12} & \tau_{13} \\ \tau_{12} & \sigma_{22} - \lambda_2 & \tau_{23} \\ \tau_{13} & \tau_{23} & \sigma_{33} - \lambda_2 \end{bmatrix} \underbrace{\begin{bmatrix} n_{12} \\ n_{22} \\ n_{32} \end{bmatrix}}_{[\underline{n}_2]} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Terminology:

λ_2 – 2nd principal stress , $[\underline{n}_2]$ – 2nd principal plane

Substitute $\lambda = \lambda_2$ and solve for $\underline{n}_3$ ← eigenvector associated with eigenvalue λ_3

$$\begin{bmatrix} \sigma_{11} - \lambda_3 & \tau_{12} & \tau_{13} \\ \tau_{12} & \sigma_{22} - \lambda_3 & \tau_{23} \\ \tau_{13} & \tau_{23} & \sigma_{33} - \lambda_3 \end{bmatrix} \underbrace{\begin{bmatrix} n_{13} \\ n_{23} \\ n_{33} \end{bmatrix}}_{[\underline{n}_3]} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Terminology:

λ_3 – 3rd principal stress , $[\underline{n}_3]$ – 3rd principal plane

Properties of Principal Planes at a point

Principal planes are the planes on which the normal component of traction is maximum/minimum. The normals of principal planes turn out to be the eigenvectors of the stress tensor.

As the stress matrix is a 3×3 matrix $\begin{cases} \to 3 \text{ eigenvalues} \\ \to 3 \text{ eigenvectors} \end{cases}$

Are these eigenvalues and eigenvectors REAL-valued?

Recall that, for symmetric matrices, eigenvalues are always REAL-VALUED. So are the eigenvectors.

1) If $\lambda_1 \neq \lambda_2 \neq \lambda_3$ (distinct eigenvalues)

The associated eigenvectors are unique and they are perpendicular to each other

$$\underline{n}_1 \perp \underline{n}_2 \perp \underline{n}_3 \quad \text{(Prove in tutorial 4)}$$

2) If $\lambda_1 = \lambda_2 \neq \lambda_3$ (two eigenvalues repeat)

Only $\underline{n}_3$ is unique and every direction perpendicular to the $\underline{n}_3$ direction is a principal direction

(Prove in Tutorial 4)

3) If $\lambda_1 = \lambda_2 = \lambda_3$ (all three eigenvalues repeat)

Then every direction is a principal direction

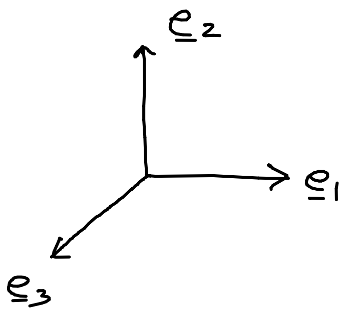
Representation of stress tensor in the coordinate system of its eigenvectors

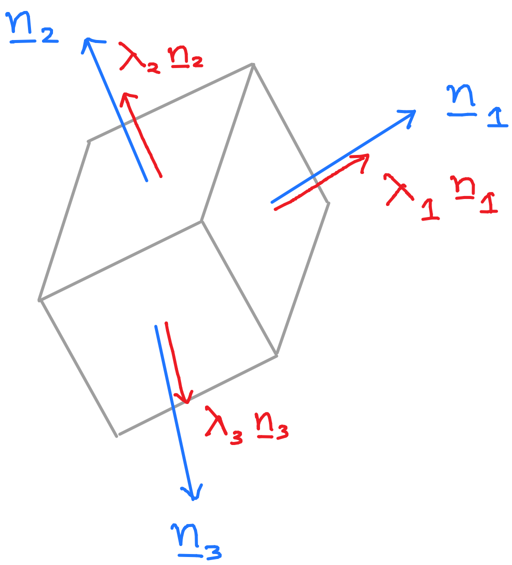
Let us choose three perpendicular eigenvectors to be the basis vectors of a coordinate system and then represent the stress tensor in this coordinate system.

By definition, the traction on the principal planes will simply be $\lambda \underline{n}$ (no shear components would be present)

The stress matrix will become diagonal when expressed in the coordinate system spanned by principal directions

$$[\underline{\underline{\sigma}}]_{\begin{pmatrix}\underline{n}_1\\ \underline{n}_2\\ \underline{n}_3\end{pmatrix}} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$





With a cuboid element's faces along the principal directions there will be no shear component and only normal components λ_1, λ_2, λ_3 will be present