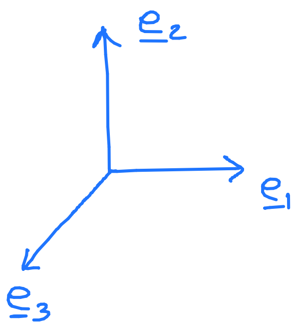
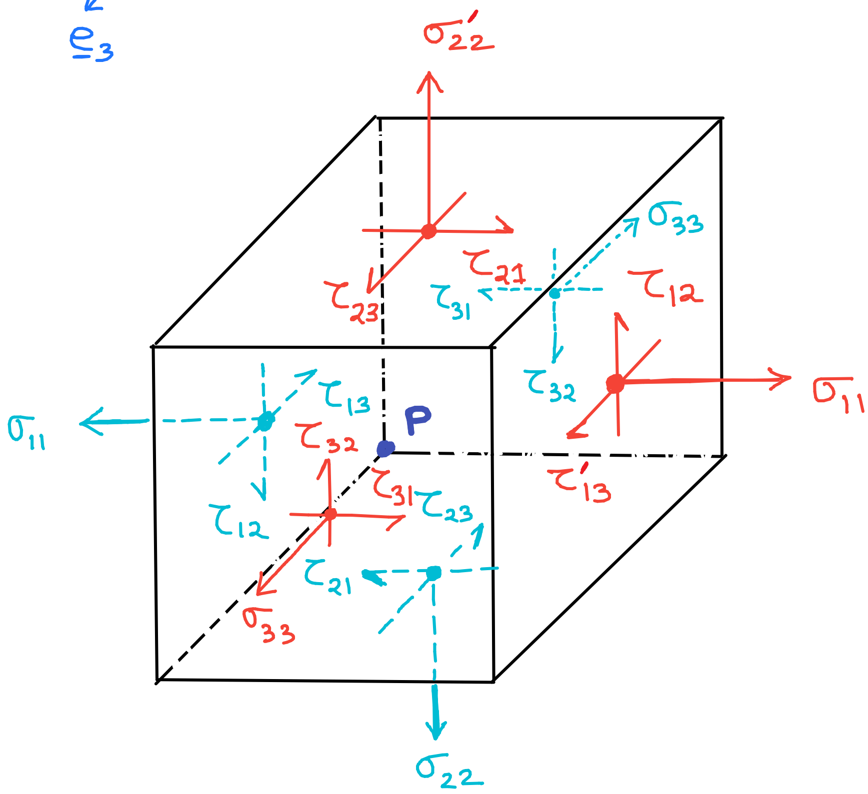
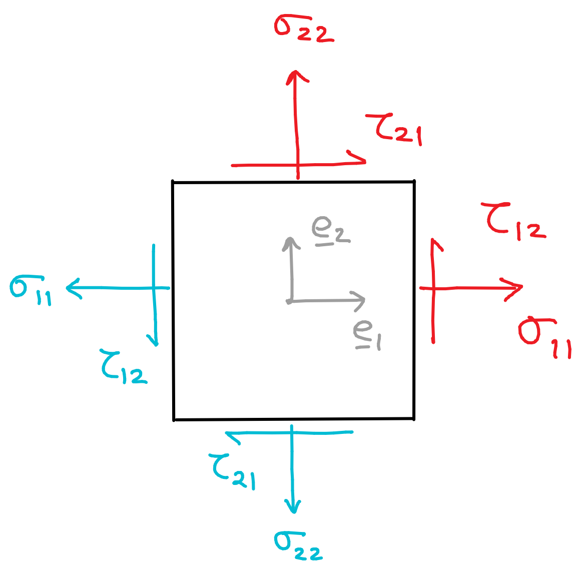
Representation of stress components



If we take out a small cuboid with origin at point P inside the body, the state of stress of the cuboid can be depicted as below:



3D state of stress

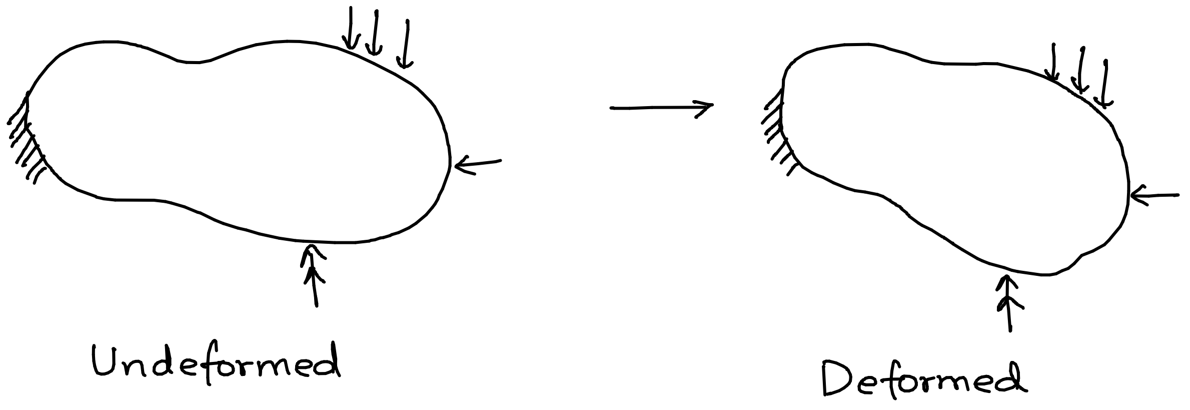


2D/ Plane state of stress

- The cuboid represents a small volume element from inside the body
- Note that all stress components shown are positive
  - Positively directed force component act on +ve plane
  - Negatively directed force component act on −ve plane
- The stress components are assumed to be uniform over the faces

STRESS EQUILIBRIUM EQUATIONS

If you apply load on a body, the body gets deformed.



You may be interested in :

1. final deformed configuration of the body

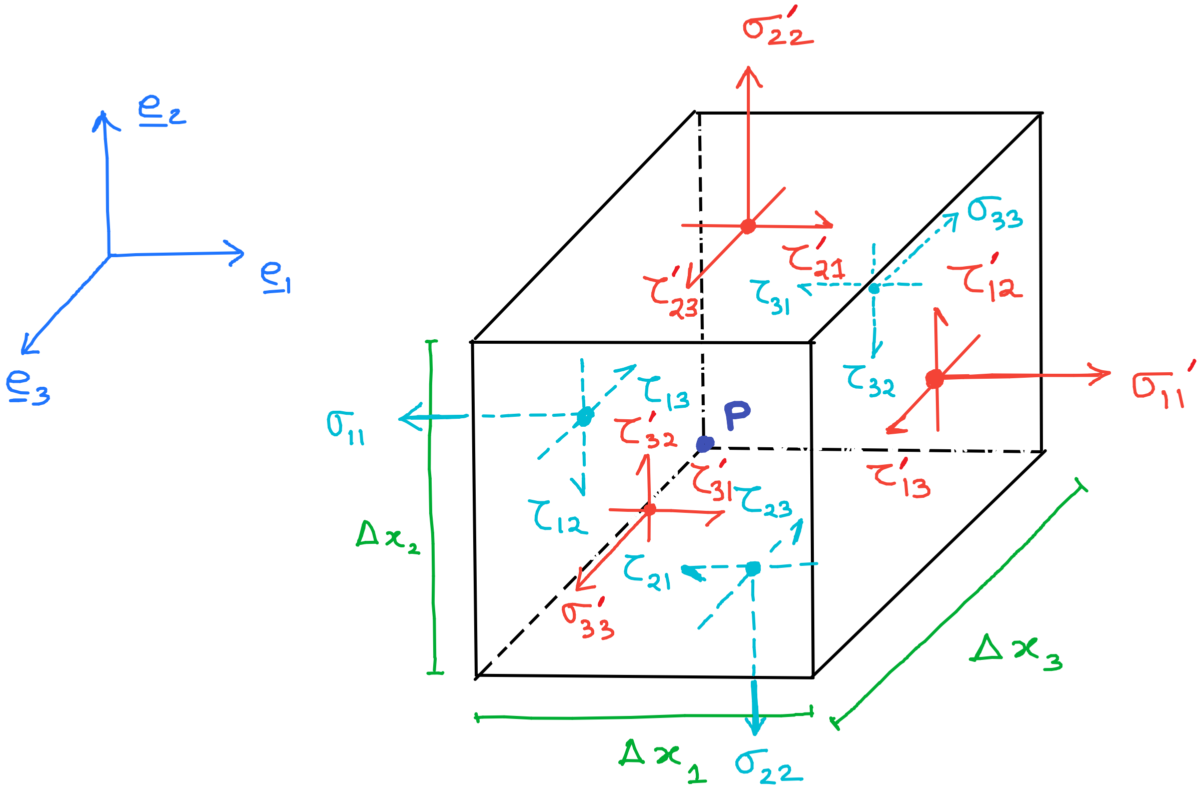
→ 2. the distribution of stress tensor in the body (since stress tensor vary within the body from pt to pt

To know where the body can fail, we would need to know the stress components at every point in the body.

How to obtain the distribution of stress tensor in the body?

We can obtain the distribution of stress tensor from the "stress-equilibrium equations" and ultimately this distribution will help in designing against failure. Lets derive these equations

We begin by taking an infinitesimal cuboidal region in the body



- Consider an infinitesimal cuboid with point P at one corner. We will assume average traction vector acting on the faces

- The stress components on faces at distances Δx_1, Δx_2, and Δx_3 ar are different from the components at the faces passing through point P since these planes donot pass through P

- How are σ'_{11}, τ'_{12}, τ'_{13} related to σ_{11}, τ_{12}, τ_{13} ?

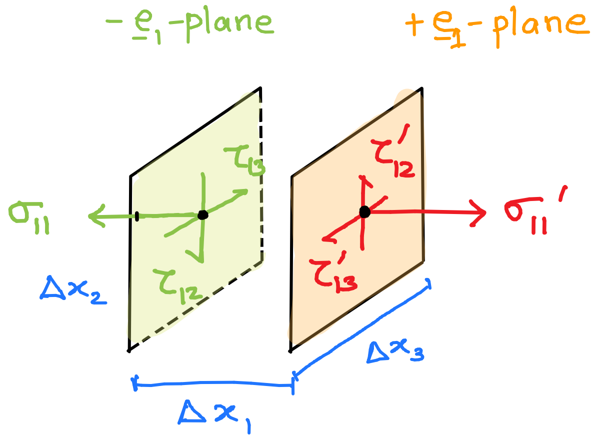
  Use Taylor series expansion,

$$\sigma'_{11} = \sigma_{11}(x_1 + \Delta x_1) = \sigma_{11}(x) + \frac{\partial \sigma_{11}}{\partial x_1} \Delta x_1 + \text{higher order terms}$$

$$\tau'_{12} = \tau_{12}(x_1 + \Delta x_1) = \tau_{12}(x) + \frac{\partial \tau_{12}}{\partial x_1} \Delta x_1 + \cdots$$

$$\tau'_{13} = \tau_{13}(x_1 + \Delta x_1) = \tau_{13}(x) + \frac{\partial \tau_{13}}{\partial x_1} \Delta x_1 + \cdots$$

Similarly, we can express relations for σ'_{22}, τ'_{21}, τ'_{23}, σ'_{33}, τ'_{32}, τ'_{31}



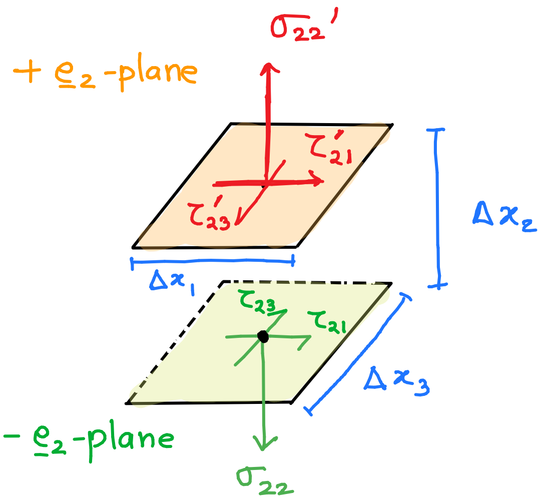
Stress components on right face

$$\sigma_{11} + \frac{\partial \sigma_{11}}{\partial x_1}\Delta x_1, \quad \tau_{12} + \frac{\partial \tau_{12}}{\partial x_1}\Delta x_1, \quad \tau_{13} + \frac{\partial \tau_{13}}{\partial x_1}\Delta x_1$$

Stress components on left face

$$\sigma_{11}, \quad \tau_{12}, \quad \tau_{13}$$

Area of both faces : $\Delta x_2 \, \Delta x_3$



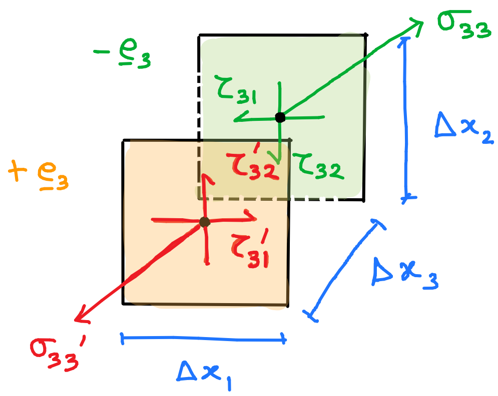
Stress components on top face

$$\sigma_{22} + \frac{\partial \sigma_{22}}{\partial x_2}\Delta x_2, \quad \tau_{21} + \frac{\partial \tau_{21}}{\partial x_2}\Delta x_2, \quad \tau_{23} + \frac{\partial \tau_{23}}{\partial x_2}\Delta x_2$$

Stress components on left face

$$\sigma_{22}, \quad \tau_{21}, \quad \tau_{23}$$

Area of both faces : $\Delta x_1 \, \Delta x_3$



Stress components on front face

$$\sigma_{33} + \frac{\partial \sigma_{33}}{\partial x_3}\Delta x_3, \quad \tau_{31} + \frac{\partial \tau_{31}}{\partial x_3}\Delta x_3, \quad \tau_{32} + \frac{\partial \tau_{32}}{\partial x_3}\Delta x_3$$

Stress components on rear face

$$\sigma_{33}, \quad \tau_{31}, \quad \tau_{32}$$

Area of both faces : $\Delta x_1 \, \Delta x_2$

If a continuous body is in equilibrium, then any isolated part of the body must also be in equilibrium. The requirements of eq^m implies that certain conditions must be satisfied by the stress components

center of cuboid

For equilibrium, $\quad \sum \underline{M}_o = 0 \quad$ and $\quad \sum \underline{F} = 0$

$\sum M_o = 0$ about $\underline{e}_3$-axis

$$\Rightarrow -\left(\tau_{21}' \Delta x_1 \Delta x_3\right) \frac{\Delta x_2}{2} - \left(\tau_{21} \Delta x_1 \Delta x_3\right) \frac{\Delta x_2}{2}$$

$$+ \left(\tau_{12}' \Delta x_2 \Delta x_3\right) \frac{\Delta x_1}{2} + \left(\tau_{12} \Delta x_2 \Delta x_3\right) \frac{\Delta x_1}{2} = 0$$

$$\Rightarrow \left[\tau_{12} + \frac{\partial \tau_{12}}{\partial x_1} \Delta x_1 + \tau_{12} - \tau_{21} - \frac{\partial \tau_{21}}{\partial x_2} \Delta x_2 - \tau_{21}\right] \frac{\Delta x_1 \Delta x_2 \Delta x_3}{2} = 0$$

$$\Rightarrow \quad 2\tau_{12} - 2\tau_{21} + \frac{\partial \tau_{12}}{\partial x_1} \Delta x_1 - \frac{\partial \tau_{21}}{\partial x_2} \Delta x_2 = 0$$

As Δx_1, Δx_2 and $\Delta x_3 \to 0$

$$\boxed{\tau_{12} = \tau_{21}}$$

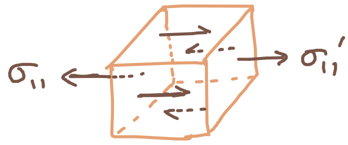
Similarly, taking moments about the center of the cuboid along $\underline{e}_1$- and $\underline{e}_2$-directions, we get :

$\sum M_o = 0 \quad$ (about $\underline{e}_2$-direction) $\qquad \boxed{\tau_{13} = \tau_{31}}$

$\sum M_o = 0 \quad$ (about $\underline{e}_1$-direction) $\qquad \boxed{\tau_{23} = \tau_{32}}$

Shear Stress components on perpendicular faces are equal in magnitude

Now applying force equilibrium



$$\xrightarrow{+} \sum F_x = 0$$

$$\Rightarrow \sigma_{11}' \Delta x_2 \Delta x_3 - \sigma_{11} \Delta x_2 \Delta x_3 + \tau_{21}' \Delta x_1 \Delta x_3 - \tau_{21} \Delta x_1 \Delta x_3$$
$$+ \tau_{31}' \Delta x_1 \Delta x_2 - \tau_{31} \Delta x_1 \Delta x_2 + \gamma_1 (\Delta x_1 \Delta x_2 \Delta x_3) = 0$$

$$\Rightarrow \left(\cancel{\sigma_{11}} + \frac{\partial \sigma_{11}}{\partial x_1} \Delta x_1\right) \Delta x_2 \Delta x_3 - \cancel{\sigma_{11} \Delta x_2 \Delta x_3}$$
$$+ \left(\cancel{\tau_{21}} + \frac{\partial \tau_{21}}{\partial x_2} \Delta x_2\right) \Delta x_1 \Delta x_3 - \cancel{\tau_{21} \Delta x_1 \Delta x_3}$$
$$+ \left(\cancel{\tau_{31}} + \frac{\partial \tau_{31}}{\partial x_3} \Delta x_3\right) \Delta x_1 \Delta x_2 - \cancel{\tau_{31} \Delta x_1 \Delta x_2}$$
$$+ \gamma_1 \Delta x_1 \Delta x_2 \Delta x_3 = 0$$

$$\Rightarrow \left(\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \tau_{21}}{\partial x_2} + \frac{\partial \tau_{31}}{\partial x_3} + \gamma_1\right) \Delta x_1 \Delta x_2 \Delta x_3 = 0$$

Dividing by $\Delta x_1 \Delta x_2 \Delta x_3$, we get

$$\Rightarrow \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \tau_{21}}{\partial x_2} + \frac{\partial \tau_{31}}{\partial x_3} + \gamma_1 = 0$$

Similarly,

$$+\uparrow \sum F_y = 0 \quad \Rightarrow \quad \frac{\partial \tau_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \tau_{32}}{\partial x_3} + \gamma_2 = 0$$

$$+\swarrow \sum F_z = 0 \quad \Rightarrow \quad \frac{\partial \tau_{13}}{\partial x_1} + \frac{\partial \tau_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + \gamma_3 = 0$$

While deriving the stress equilibrium equations, we shrink the volume of the cuboid to a point so that these equations hold at every point in the body

STRESS EQUILIBRIUM RELATIONS

Three force eq^m + Three moment eq^m $\Rightarrow$ Total SIX relations

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \tau_{21}}{\partial x_2} + \frac{\partial \tau_{31}}{\partial x_3} + \gamma_1 = 0 \qquad \tau_{21} = \tau_{12}$$

$$\frac{\partial \tau_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \tau_{32}}{\partial x_3} + \gamma_2 = 0 \qquad \tau_{31} = \tau_{13}$$

$$\frac{\partial \tau_{13}}{\partial x_1} + \frac{\partial \tau_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + \gamma_3 = 0 \qquad \tau_{23} = \tau_{32}$$

A set of PDEs

In indicial notation, one can write

$$\sum_{i=1}^{3} \frac{\partial \underline{\underline{\sigma}}}{\partial x_i} \underline{e}_i + \underline{\gamma} = \underline{0} \quad \leftarrow \text{Force eq}^m$$

$$\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^T \quad \leftarrow \text{Moment eq}^m$$

where the representation of the stress tensor $\underline{\underline{\sigma}}$ in $(\underline{e}_1 - \underline{e}_2 - \underline{e}_3)$ coordinate system will be

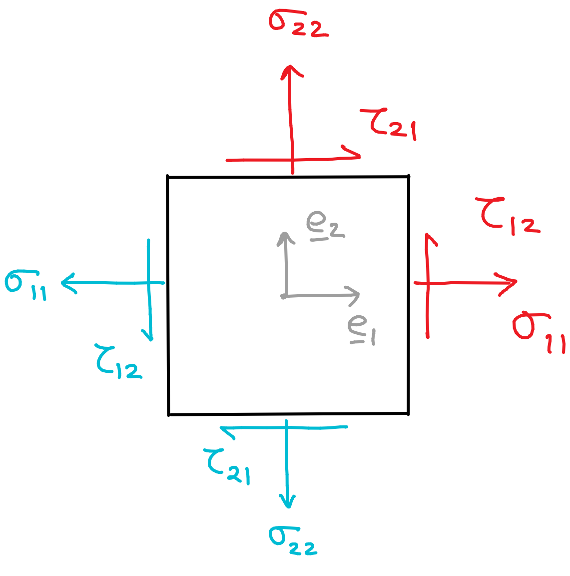
$$\left[\underline{\underline{\sigma}}\right]_{(\underline{e}_1 - \underline{e}_2 - \underline{e}_3)} = \begin{bmatrix} \sigma_{11} & \tau_{21} & \tau_{31} \\ \tau_{12} & \sigma_{22} & \tau_{32} \\ \tau_{13} & \tau_{23} & \sigma_{33} \end{bmatrix}$$

For PLANE STRESS

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \tau_{21}}{\partial x_2} + \gamma_1 = 0$$

τ_{21}

$$\frac{\partial \tau_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \gamma_2 = 0$$



The equilibrium equations for a deformable solid body

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \tau_{21}}{\partial x_2} + \frac{\partial \tau_{31}}{\partial x_3} + \gamma_1 = 0 \qquad \tau_{21} = \tau_{12}$$

$$\frac{\partial \tau_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \tau_{32}}{\partial x_3} + \gamma_2 = 0 \qquad \tau_{31} = \tau_{13}$$

$$\frac{\partial \tau_{13}}{\partial x_1} + \frac{\partial \tau_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + \gamma_3 = 0 \qquad \tau_{23} = \tau_{32}$$

The above equations are partial differential equations, and to solve them, we would need BOUNDARY CONDITIONS

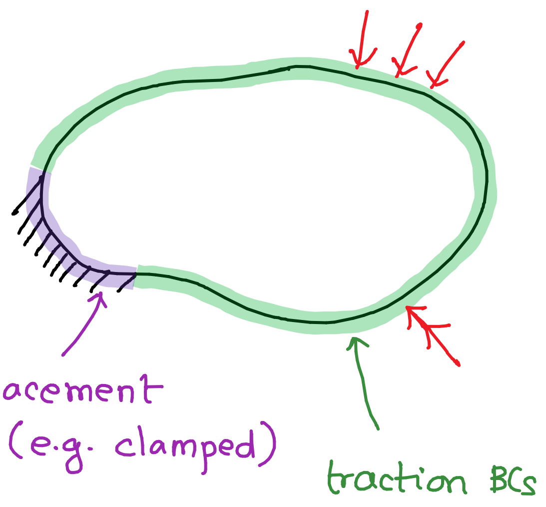
There are two types of boundary conditions

→ displacement BCs

specifies displacements on the boundary (will cover later)

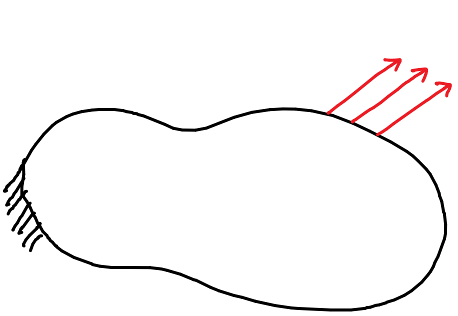
→ traction / force BCs

specifies external loads applied on the boundary (will look at now)



Relation between external load on body's surface to stress tensor

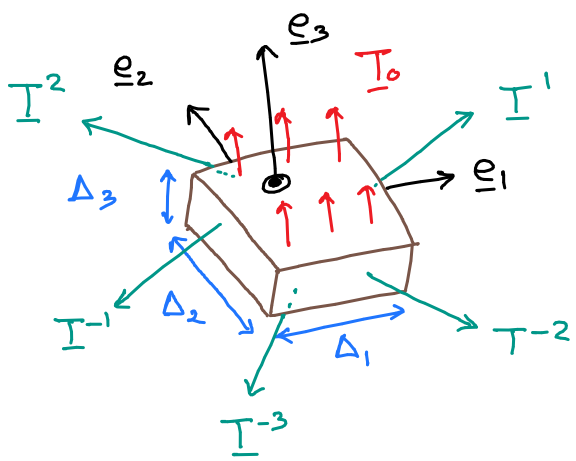
Suppose we have an arbitrary body which is clamped at some part of the boundary and a load is applied on some other part of the boundary by an external agent



Lets take a small piece from the surface. Note the a al boundary is curved, bu if we take a tiny piece, it will be almost flat

What are the forces acting on the faces of the tiny piece?

- External force (acting on the exposed face)
- Tractions (acting on the internal faces)
- Body force



From force equilibrium of this tiny piece, the total force = 0

$$\underbrace{\underline{T}^{\circ}\Delta_1\Delta_2}_{\text{External}} + \left(\underline{T}^2 + \underline{T}^{-2}\right)\Delta_1\Delta_3 + \left(\underline{T}^1 + \underline{T}^{-1}\right)\Delta_2\Delta_3$$

$$+ \underline{T}^{-3}\Delta_1\Delta_2 + \underbrace{\underline{\gamma}\,\Delta_1\Delta_2\Delta_3}_{\text{body force}} = 0$$

If you divide the above equation by $\Delta_1 \Delta_2$ and then $\Delta_3 \longrightarrow 0$, that is

$$\lim_{\Delta_3 \to 0} \frac{\underline{T}^0 \Delta_1 \Delta_2 + (\underline{T}^2 + \underline{T}^{-2}) \Delta_1 \Delta_3 + (\underline{T}^1 + \underline{T}^{-1}) \Delta_2 \Delta_3 + \underline{T}^{-3} \Delta_1 \Delta_2 + \underline{\gamma} \Delta_1 \Delta_2 \Delta_3}{\Delta_1 \Delta_2} = \underline{0}$$

$$\Rightarrow \lim_{\Delta_3 \to 0} \left[\underline{T}^0 + (\underline{T}^2 + \underline{T}^{-2}) \frac{\Delta_3}{\Delta_2} + (\underline{T}^1 + \underline{T}^{-1}) \frac{\Delta_3}{\Delta_1} + \underline{T}^{-3} + \underline{\gamma} \Delta_3 = \underline{0} \right]$$

As we take $\Delta_3 \to 0$, we are shrinking the height by pushing the bottom surface towards the top surface while keeping the surface area $\Delta_1 \Delta_2$ constant. The terms containing Δ_3 will vanish in this limit and we get $\boxed{\underline{T}^{-3} = -\underline{T}^0}$

As $\underline{T}^3$ and $\underline{T}^{-3}$ form an action and reaction pair, we obtain

$$\boxed{\underline{T}^3 = \underline{\underline{\sigma}}\, \underline{e}_3 = \underline{T}^0} \xrightarrow{\text{In general}} \boxed{\underline{\underline{\sigma}}\, \underline{n} = \underline{T}^0}$$

outward normal

This relation implies that the internal traction that generates in a body at its surface point and on an internal section that is parallel to local surface plane of the body is equal to the externally applied distributed load

This form is used as boundary condition for solving the stress equilibrium equation and is called TRACTION BOUNDARY CONDITION. The above result might appear intuitive and trivial, but we cannot conclude $\underline{T}^0 = \underline{T}^3$ trivially. We won't be able to say anything about $\underline{T}^1$ or $\underline{T}^2$ at the surface point

Now what can you say about the stress matrix on the surface

For example, if $\underline{T}^3 = \underline{T}^0$, how will the $[\underline{\underline{\sigma}}]$ matrix look?

In a chosen coordinate system $(e_1 - e_2 - e_3)$,

$$[\underline{\underline{\sigma}}]_{(e_1-e_2-e_3)} = \begin{bmatrix} \times & \times & T_1^0 \\ \times & \times & T_2^0 \\ \underbrace{T_1^0 \quad T_2^0}_{\text{by sym.}} & & T_3^0 \end{bmatrix}$$