Conservation of energy

In previous lecture, we introduce two concepts:

External work by a gradually applied load to a body causing deformation along the load

U_e

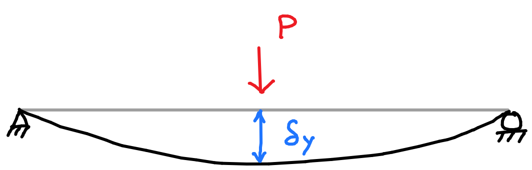
Internal stored strain energy caused by normal and shear stresses

U_i

The conservation of energy for a body mathematically implies:

$$\boxed{U_e = U_i}$$

So let's say if we wanted to find the vertical displacement δ_y of a beam under the gradual application of a load P



External work, $U_e = \frac{1}{2} P \delta_y$

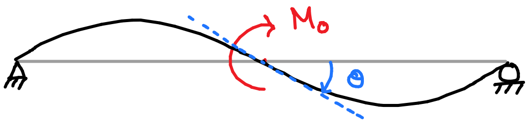
Internal strain energy would be caused by internal shear & bending moment caused by P.

$$U_i = \int_0^L \frac{M^2(x)}{2EI_z}\, dx + \int_0^L \frac{V^2(x)}{KGA}\, dx$$

By conservation of energy, $U_e = U_i$

$$\Rightarrow \quad \frac{1}{2} P \delta_y = \int_0^L \frac{M^2(x)}{2EI_z}\, dx + \int_0^L \frac{V^2(x)}{KGA}\, dx$$

If the beam was instead subjected to an external moment M_o, the moment would have caused a rotation θ at its point of application.

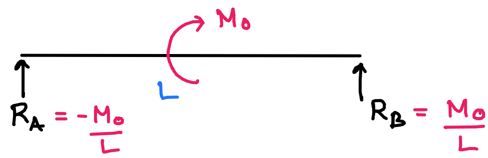


External work, $U_e = \frac{1}{2} M_o \theta$

From conservation of energy,

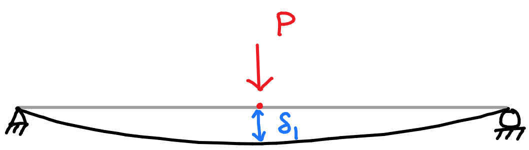
$$\frac{1}{2} M_o \theta = \int_0^L \frac{M^2(x)}{2EI} dx + \int_0^L \frac{V^2(x)}{2kGA} dx$$

Would shear force be present in the beam when only M_o is applied? Yes!



Using conservation of energy, one can find deflection or slope of a beam, or deformation of any body in general. However, the relation has very limited use because this method can be used to find deformation only when a single load is acting.

For more than one external load (force/moment), the external work for each loading would have its associated own unknown displacement. As such none of the displacements can be determined using a single equation $U_e = U_i$.



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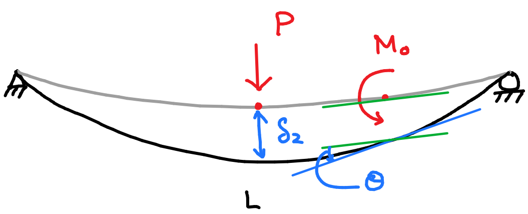


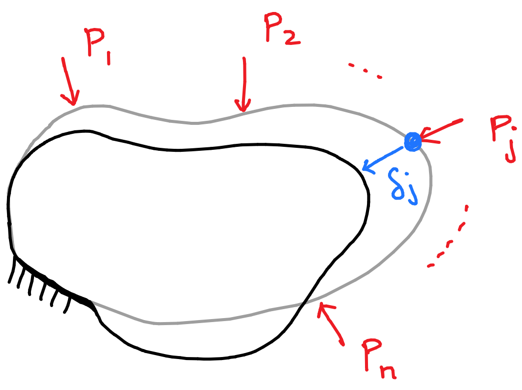
can be determined

$$\frac{1}{2} P \delta_1 + P \delta_2 + \frac{1}{2} M_o \theta = \int_0^L \frac{M^2(x)}{2EI} dx + \int_0^L \frac{V^2(x)}{2kGA} dx$$

2 unknowns, 1 eqn

Castigliano's Theorem

This theorem provides a way to determine displacement and rotation at a point in a body even under the application multiple loads.



An arbitrary body subjected to a series of forces $P_1, P_2, .., P_j, ..$ P_n will cause external work, U_e.

This external work must be equal to the internal strain energy stored in the body, U_i

$$U_i = U_e\,(P_1, P_2, \ldots, P_j, \ldots, P_n)$$

Now, if any one of the external forces, say P_j, is increased by a differential amount dP_j, the internal stored energy will also increase:

$$U_i + dU_i = U_i + \frac{\partial U_i}{\partial P_j}\, dP_j$$

Due to an increase dP_j, the body at the point of action of dP_j will displace by a differential amount $d\delta_j$ in the direction of dP_j. The increment of strain energy would then be:

$$dU_i = \underbrace{\frac{1}{2}\, \underset{\text{small}}{dP_j}\, \underset{\text{small}}{d\delta_j}}_{\text{can be ignored}} + dP_j\, \delta_j$$

δ_j: total displacement along dP_j caused by deflections due to $P_1, \ldots, P_n$ being applied

Therefore, we have:

$$dU_i = dP_j \, \delta_j$$

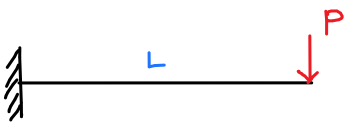
$$\Rightarrow \boxed{\delta_j = \frac{dU_i}{dP_j}}$$

← Castigliano's theorem states that the displacement at a pt in the body is equal to the first derivative of the strain energy in the body w.r.t. a force acting at that point and along the direction of the displacement.

Using a dummy force/moment

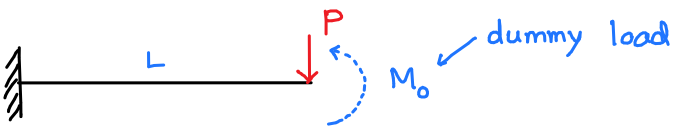
If we are interested in find displacement at a point in the body where there is no corresponding applied load, a dummy load is introduced and then Castigliano's theorem is applied.

Ex1: Find the rotation of the free end of the cantilever beam



Note that the corresponding load for getting rotation at the end is a moment applied at the end

1. Apply a dummy load P_0 or M_0 the location of required deflection

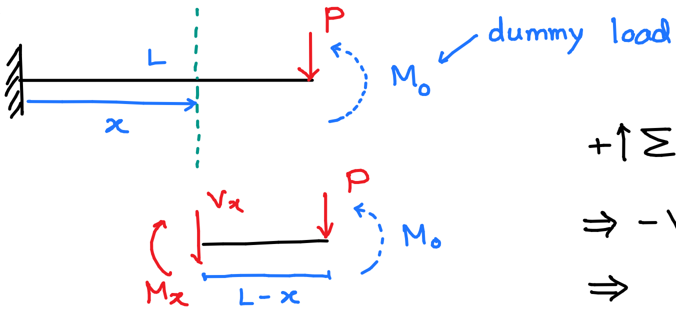


2. Obtain the strain energy of the body, U_i

3. Apply Castigliano's theorem, i.e. $\frac{\partial U_i}{\partial P_0}$ or $\frac{\partial U_i}{\partial M_0}$

4. Finally, set $P_0 = 0$ (or $M_0 = 0$) in the expression for deflection.

Ex 1



$$+\uparrow \sum F_y = 0$$

$$\Rightarrow -V(x) - P = 0$$

$$\Rightarrow V(x) = -P$$

$$\circlearrowleft + \sum M_{\text{left end}} = 0$$

$$\Rightarrow -M(x) - P(L-x) + M_o = 0$$

$$\Rightarrow M(x) = M_o - P(L-x)$$

The axial force and torque are zero. So the strain energy would be due to bending moment and shear force.

$$U_i = \int_0^L \frac{M^2(x)}{2EI}\,dx + \int_0^L \frac{V^2(x)}{2kGA}\,dx$$

$$= \int_0^L \left[\frac{(M_o - P(L-x))^2}{2EI} + \frac{(-P)^2}{2kGA}\right] dx$$

$$\theta = \frac{\partial U_i}{\partial M_o} = \int_0^L \left[\frac{\partial}{\partial M_o}\frac{(M_o - P(L-x))^2}{2EI} + \frac{\partial}{\partial M_o}\frac{P^2}{2kGA}\right] dx$$

$$= \int_0^L \frac{\cancel{M_o}^{\,0} - P(L-x)}{EI}\,dx$$

$$= -\frac{P}{EI}\int_0^L (L-x)\,dx = -\frac{P}{EI}\left(Lx - \frac{x^2}{2}\right)\Big|_0^L = -\frac{PL^2}{2EI}$$

the direction of rotation would be opposite of the direction of M_o assumed.