Energy Methods

In the last lecture, we talked about deflection of beams subjected to transverse loads and moments. However, often, we may be interested not in the full deflection profile but only in the deflection at a specific point in the beam. For such cases, energy methods offers an easier alternative to finding at certain points.

External Work and Strain Energy

We first define the work caused by an external force/moment and show how to express this work in terms of a body's strain energy.

Work done by a force/moment

A force does work when the force acting at a point on the body undergoes a displacement in the same direction as the force

$$dU_e = F\,dx$$

displacement in the same direction as the force

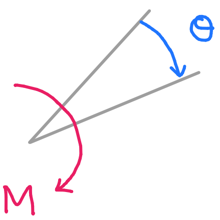
work done

If the total displacement is δ, then the external work done is:

$$U_e = \int_0^{\delta} F\,dx$$

Similarly, a moment M does work when it causes a rotation $d\theta$ along its line of action. The external work done by moment due to a total rotation of θ is:

$$U_e = \int_0^{\theta} M\,d\theta$$

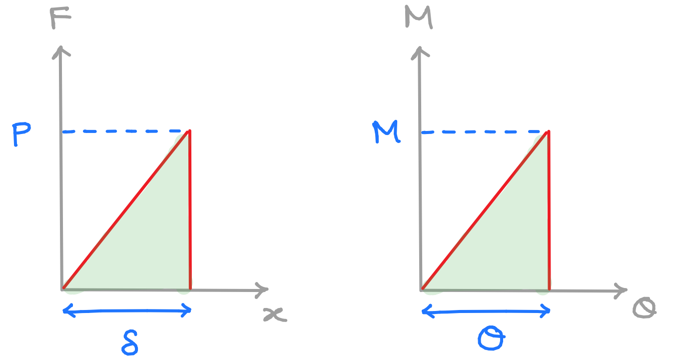


If a force/moment is gradually applied i.e. increased from zero to some value P (or M), with the corresponding deformation (displacement / rotation) increasing from 0 to a final value, then the work done in the case of linear elastic material:

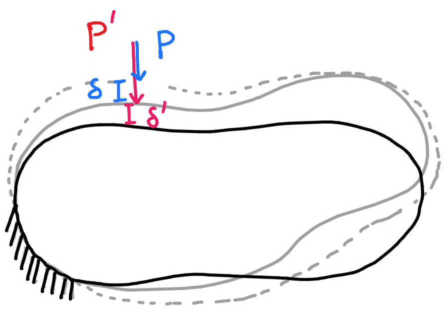
$$U_e = \frac{1}{2} F \delta$$

or,

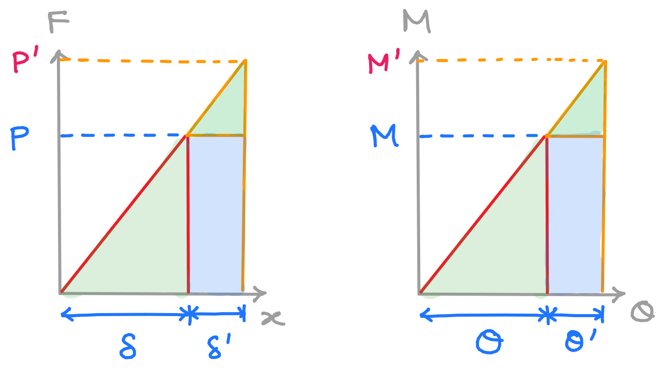
$$U_e = \frac{1}{2} M \theta$$



Now, suppose a force P is already applied to the body at a point and another force P′ is applied gradually at the same point such that the body is displaced further by δ', then the additional work done by P is



$$U_e' = P \delta'$$



Similarly, for a moment that is already applied to the body and other loadings further rotate the body by an amt θ', then the additional work done is : $U_e' = M \theta'$

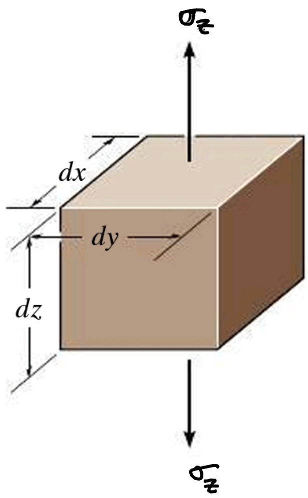
Strain Energy stored in a body

When loads (forces/moments) are applied to a body, they will deform the material, and if no energy is lost in the form of heat or sound, the external work done by the loads will be converted into internal stored energy called the strain energy.

Strain energy is stored in the body and is caused by action of either normal or shear stress.

Strain energy due to NORMAL STRESS

To obtain strain energy caused by a normal stress, say σ_z, consider the volume element



The force created on element's top & bottom surfaces:

$$dF_z = \sigma_z \, dx \, dy$$

If this force is applied gradually and the corresponding displacement in the same direction as σ_z is $d\delta_z = \epsilon_z \, dz$, then work done by dF_z is

$$dU_i = \frac{1}{2} dF_z \, d\delta_z = \frac{1}{2}(\sigma_z \, dx dy)(\epsilon_z \, dz)$$

$$= \frac{1}{2} \sigma_z \epsilon_z \underbrace{dx \, dy \, dz}_{dV}$$

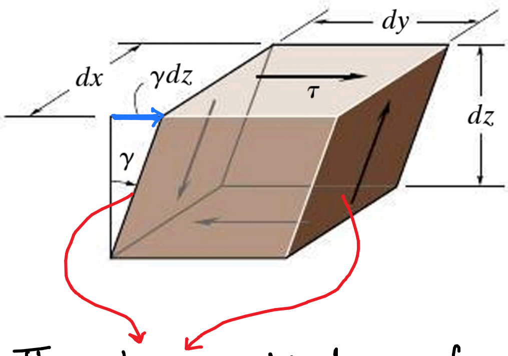
This is called the internal stored strain energy. It is always positive.

volume of the element

For a body of finite size, the strain energy in the body is

$$U_i = \int_V \frac{\sigma \epsilon}{2} dV \xrightarrow[\text{elastic material}]{\text{linear}} \boxed{U_i = \frac{1}{2}\int_V \sigma\left(\frac{\sigma}{E}\right) dV = \frac{1}{2}\int_V \frac{\sigma^2}{2E} dV}$$

Strain energy due to SHEAR STRESS



Here the force acting on the top

$$dF = \tau \, dx \, dy$$

causing the surface to be displaced by γdz relative to bottom surface

The two vertical surfaces only rotate and therefore the shear forces on these faces do no work. Hence, strain energy stored in the volume element becomes

$$dU_i = \frac{1}{2}(\tau \, dx \, dy)(\gamma \, dz)$$

$$= \frac{1}{2}\tau\gamma \underbrace{dx \, dy \, dz}_{dV}$$

$$= \frac{1}{2}\tau\gamma \, dV$$

Therefore, strain energy stored in a body subjected to shear stress:

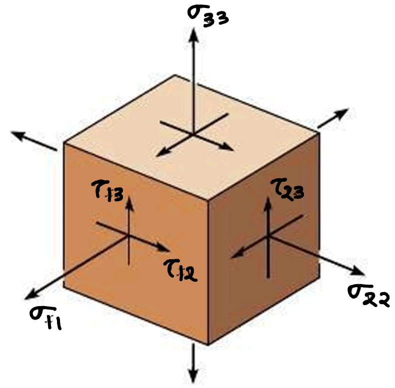
$$U_i = \frac{1}{2}\int_V \tau\gamma \, dV$$

For linear elastic materials : $\gamma = \dfrac{\tau}{G}$ and

$$\boxed{U_i = \int_V \frac{\tau^2}{2G} \, dV}$$

For general state of stress, the total strain energy in a body becomes

$$U_i = \int_V \frac{1}{2}\left[\sigma_{11}\epsilon_{11} + \sigma_{22}\epsilon_{22} + \sigma_{33}\epsilon_{33} + \tau_{12}\gamma_{12} + \tau_{23}\gamma_{23} + \tau_{13}\gamma_{13}\right] dV$$

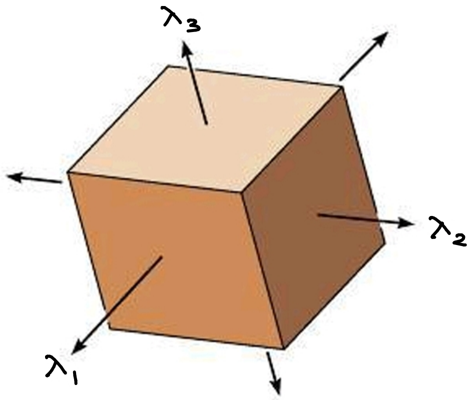


For linear elastic materials:

$$U_i = \int_V \left[\frac{1}{2E}\left(\sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2\right) - \frac{\nu}{E}\left(\sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{11}\sigma_{33}\right) + \frac{1}{2G}\left(\tau_{12}^2 + \tau_{23}^2 + \tau_{13}^2\right)\right] dV$$

If one considers strain energy in coordinates given by principal directions, then the expression for strain energy reduces to a simpler form:

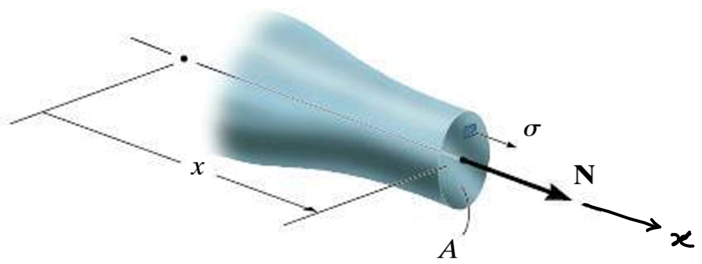
$$U_i = \int_V \left[\frac{1}{2E}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2\right) - \frac{\nu}{E}\left(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1\right)\right] dV$$



Elastic strain energy for various types of loading

Using the expressions developed for elastic strain energy, we will now write down strain energy stored in a body when a member is subjected to an axial load, bending moment, transverse shear, and torsional moment.

Axial load



Stress $\sigma_x = \dfrac{N(x)}{A}$

Strain energy,

$$U_i = \int_V \frac{\sigma^2}{2E}\, dV$$

$$= \int \frac{N_x^2}{2EA^2}\, dV$$

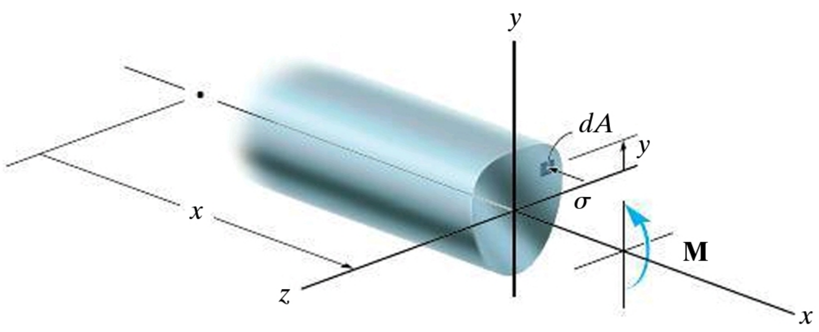
If we choose a differential segment of the bar having a volume $dV = A\,dx$, then the general formula for U_i :

$$\boxed{U_i = \int \frac{N_x^2}{2AE}\, dx}$$

Bending moment

Bending stress

$$\sigma_x = \frac{M(x)\, y}{I_z}$$



Strain energy,

$$U_i = \int_V \frac{\sigma^2}{2E}\, dV$$

$$= \int_V \frac{1}{2E}\left(\frac{M y}{I_z}\right)^2 \underbrace{dV}_{dA\,dx}$$

$$= \int_0^L \frac{M^2}{2EI_z^2}\underbrace{\left(\int_A y^2\, dA\right)}_{I_z} dx = \int_0^L \frac{M^2}{2EI_z}\, dx$$

So, strain energy due to bending is :

$$\boxed{U_i = \int_0^L \frac{M^2(x)}{2EI_z}\, dx}$$

To evaluate this strain energy, you must express the internal bending moment as a function of its position x along the beam and then perform the integration over the entire length of beam.

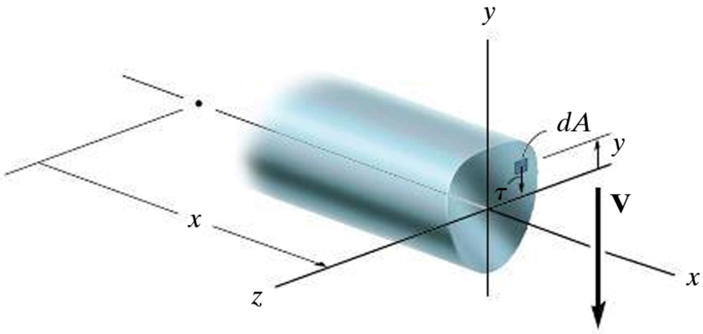
Transverse shear

In a similar fashion, one can work out the strain energy stored due to internal shear force V

$$U_i = \int_0^L \frac{V^2(x)}{2\,k\,G\,A}\,dx$$

shear correction factor / form factor

shear modulus



Torsional moment

$$U_i = \int_0^L \frac{T^2(x)}{2\,G\,J}\,dx$$

shear modulus

polar moment of inertia of C/s area