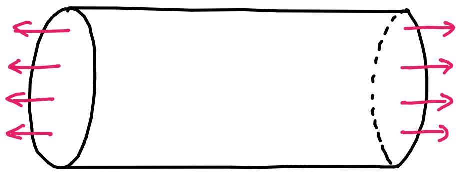
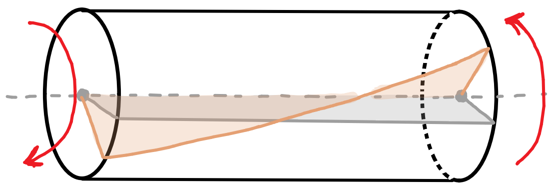
# Extension-torsion-inflation problems

If we apply axial force to a cylinder by applying traction on its two end cross-sections, the cylinder will get stretched. Such deformation is called extension



If we hold the two ends of a cylinder and rotate them in opposite directions, different cross-sections of the cylinder will get rotated by different angles. Such deformation is called twisting or torsion and requires torque of equal & opposite directions applied at the ends of the cylinder.



Finally, if we apply pressure to a hollow cylinder from within its inner cavity, the inner and outer radii of the cylinder increases. Such deformation is called inflation

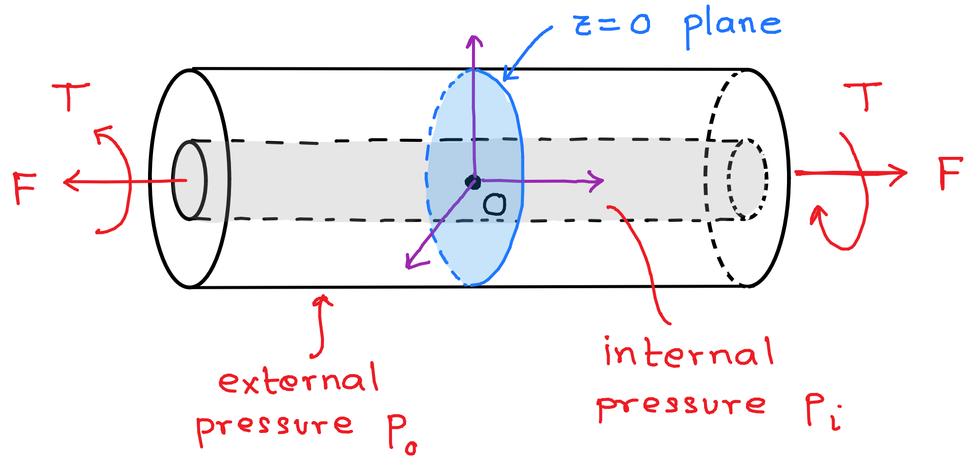
We will study the deformation of a cylinder when it is subjected to combined axial load, torque, internal & external pressure. Due to the assumption of linearity in stress-strain relations and strain-displacement relations, along with boundary conditions being linear in unknown displacement components, we obtain a linear system of differential-algebraic equations.

The linearity allows us to use the principle of superposition i.e. we can obtain solution to extension, torsion, and inflation problems individually and then combine the individual solutions to obtain solution to the combined loading problem.

(Note the superposition principle does not hold in nonlinear elasticity)

## Problem definition

Consider a hollow cylinder subjected to axial force $F$, torque $T$, and internal & external pressure $P_i$ and $P_o$




We will solve this problem using cylindrical coordinates.

We will need to find

3 disp. $\begin{cases} u_r \\ u_\theta \\ u_z \end{cases}$, 6 strains $\begin{cases} \epsilon_{rr} \\ \epsilon_{\theta\theta} \\ \epsilon_{zz} \\ \gamma_{rz} \\ \gamma_{r\theta} \\ \gamma_{\theta z} \end{cases}$, 6 stresses $\begin{cases} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{zz} \\ \tau_{rz} \\ \tau_{r\theta} \\ \tau_{\theta z} \end{cases}$

## Simplifications using certain assumptions

The displacement components are in general functions of $r, \theta, z$

$$u_r = u_r(r, \theta, z), \qquad u_\theta = u_\theta(r, \theta, z), \qquad u_z = u_z(r, \theta, z)$$

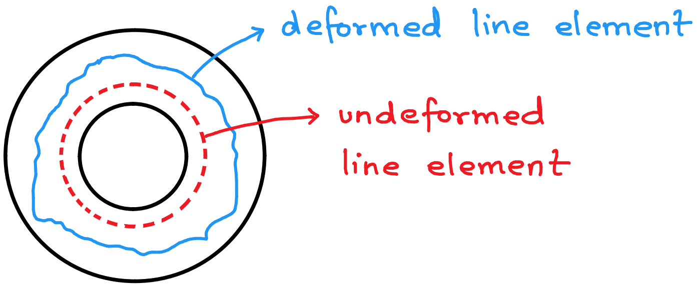
For the special cases of deformation that we are dealing with, some dependencies of the displacement components on $r$, $\theta$, and $z$ can be removed.

### (a) Axisymmetry assumption

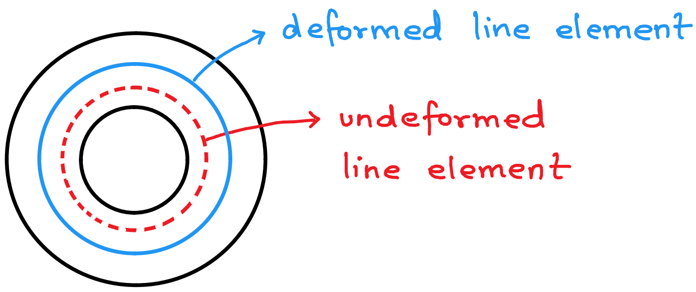
Due to axisymmetry of the loading applied to the cylinder the deformation induced would also be axisymmetric.

$\Rightarrow$ none of the displacement components depend on $\theta$

i.e. if we consider any two points in the cylinder with the same $r$ and $z$ coordinates but different $\theta$ coordinate, the displacement at the two points are SAME.




Non-axisymmetric deformation

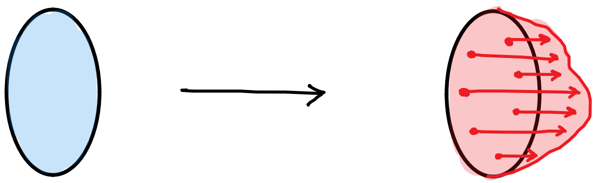



Axisymmetric deformation

$$\left.\begin{aligned} u_r(r, \theta, z) &= u_r(r, z) \\ u_\theta(r, \theta, z) &= u_\theta(r, z) \\ u_z(r, \theta, z) &= u_z(r, z) \end{aligned}\right\} \text{due to axisymmetry of loading and that in geometry of cylinder}$$

(b) No warping of cross-section

All points in a given cross-section have the same $z$ coor. The displacement component $u_z$ displaces these points in the axial direction. If $u_z$ changes with $r$, two points on such c/s (having different radial coordinate) would displace in the axial direction by different amounts. This will make the deformed c/s non-planar, which is called warping of the c/s. We assume no such warping occurs.

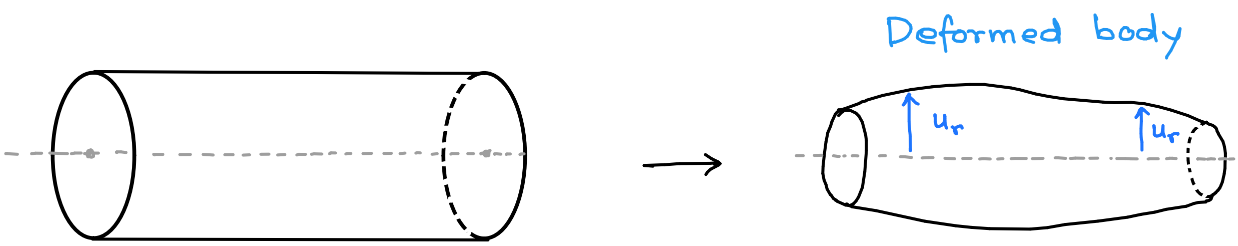


warped c/s

$$u_z(r,z) \xrightarrow{\text{plane section}} u_z(z)$$

remains plane before and after deformation

(c) Axial homogeneity




Axially non-homogeneous

If the displacement $u_r$ is independent of the axial direction $z$, then one obtains axial homogeneity. If say $u_r$ changes with $z$, the final radius of a cylinder (with initially constant radius along the axis) at different locations along the axis will be different and the deformed configuration will no longer be simple.

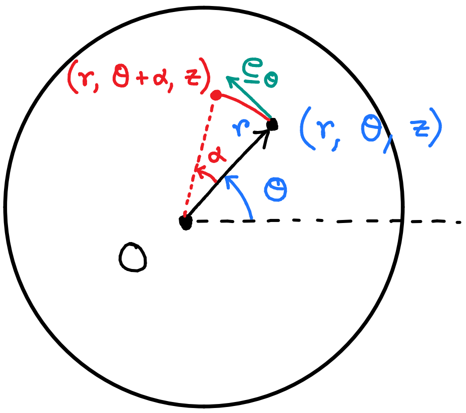
$$u_r(r,z) \xrightarrow[\text{homogeneity}]{\text{axial}} u_r(r)$$

## (d) $u_\theta$ generated only during torsion

We also have some restrictions on $u_\theta$. Both extension & inflation generates axial and radial displacements ($u_z$ & $u_r$) but no $u_\theta$ for isotropic cylinders.

An application of torque, on the other hand, causes a typical c/s to rotate which only generates $u_\theta$ for isotropic cylinders.

To quantify this displacement $u_\theta$, let us consider a c/s of the cylinder which rotates by an angle $\alpha$




A cross-section of a cylinder rotating by angle $\alpha$; a point on the c/s displaces due to this rotation from $(r, \theta, z)$ to another pt $(r, \theta+\alpha, z)$

When the arc $(r\alpha)$ is small, it would become a straight line in the $\underline{e}_\theta$ - direction. Thus, we will get

$$u_\theta = \alpha(z)\, r \qquad (\text{if } \alpha \text{ is very small})$$

In the case of torsion, $\alpha$ can be written in terms of end-to-end rotation $\Omega$. If the rightmost c/s rotates by $\frac{\Omega}{2}$ in one direction while the leftmost c/s rotates by $\frac{\Omega}{2}$ in the other dir. then

(Note: the mid c/s at $z=0$ does not rotate)

$$\boxed{\alpha(z) = \frac{\Omega}{L} z}$$

$\alpha \rightarrow$ called the twist

So we have simplified our displacements as

$$u_r = u_r(r)$$

$$u_\theta = \frac{\Omega}{L} z r$$

Only appears in the case of torsion

$$u_z = u_z(z)$$

- Simplified strain-displacement relations

$$[\underset{=}{\epsilon}]_{(r,\theta,z)} = \begin{bmatrix} \frac{\partial u_r}{\partial r} & \frac{1}{2}\left[\frac{1}{r}\left(\frac{\partial u_r}{\partial \theta} - u_\theta\right) + \frac{\partial u_\theta}{\partial r}\right] & \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) \\ & \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + u_r\right) & \frac{1}{2}\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right) \\ \text{SYM} & & \frac{\partial u_z}{\partial z} \end{bmatrix}$$

Upon using simplified representations of $u_r$, $u_\theta$ & $u_z$, we get

$$[\underset{=}{\epsilon}]_{(r,\theta,z)} = \begin{bmatrix} \overset{u_r'}{\frac{du_r}{dr}} & 0 & 0 \\ 0 & \frac{u_r}{r} & \frac{\Omega r}{2L} \\ 0 & \frac{\Omega r}{2L} & \underset{u_z'}{\frac{du_z}{dz}} \end{bmatrix}$$

this shear strain is non-zero and it arises due to torsion

- Simplified stress-strain relations

$$\sigma_{rr} = \lambda\left(u_r' + \frac{u_r}{r} + u_z'\right) + 2\mu u_r' \,, \qquad \tau_{r\theta} = 2\mu\epsilon_{r\theta} = 0$$

$$\sigma_{\theta\theta} = \lambda\left(u_r' + \frac{u_r}{r} + u_z'\right) + 2\mu \frac{u_r}{r} \,, \qquad \tau_{rz} = 2\mu\,\epsilon_{rz} = 0$$

$$\sigma_{zz} = \lambda\left(u_r' + \frac{u_r}{r} + u_z'\right) + 2\mu u_z' \,, \qquad \tau_{\theta z} = 2\mu\frac{\Omega r}{2L} = \frac{\mu\Omega r}{L}$$

- Simplified equilibrium equations

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + b_r = 0$$

$$\frac{\partial \tau_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + 2\frac{\tau_{r\theta}}{r} + b_\theta = 0$$

$$\frac{\partial \tau_{zr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{z\theta}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\tau_{rz}}{r} + b_z = 0$$

The body forces are assumed to be zero. Furthermore, most of the terms in the above equilibrium equations become zero, and the equations simplify to

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0$$

$$\frac{\partial \sigma_{zz}}{\partial z} = 0$$

## Solving the simplified equations

As $u_r$ and $u_z$ are functions of $r$ and $z$, respectively, we can rewrite $\sigma_{zz}$ as

$$\sigma_{zz} = \lambda\left(u_r' + \frac{u_r}{r} + u_z'\right) + 2\mu u_z'$$

$$= \underbrace{(\lambda + 2\mu)\, u_z'}_{\text{depends upon } z} + \underbrace{\lambda\left(u_r' + \frac{u_r}{r}\right)}_{\text{depends upon } r}$$

But from one of the equilibrium equations, we have that

$$\frac{\partial \sigma_{zz}}{\partial z} = 0 \quad \Rightarrow \quad \sigma_{zz} \text{ does not depend upon } z$$

$$\Rightarrow \quad \sigma_{zz} = f(r) \;\; \Big\} \text{ must be a function of } r \text{ only}$$

$\therefore$ The term $u_z'$ must be a constant

$$\Rightarrow u_z = c_1 z + c_2$$

Also, $u_z = 0$ at the cylinder's mid-section $(z=0)$ [disp BC]

Thereforce, $c_2 = 0 \quad \Rightarrow \quad u_z = c_1 z$