Applications of Elasticity

The isotropic linear elastic material model was introduced in the last few lectures. From now on, we will look at applications concerning isotropic linear elastic materials; they will have specific geometries and will be subjected to particular types of load. We will encounter the following applications:

(1) Solutions of thick-walled cylinders under extension - torsion - inflation

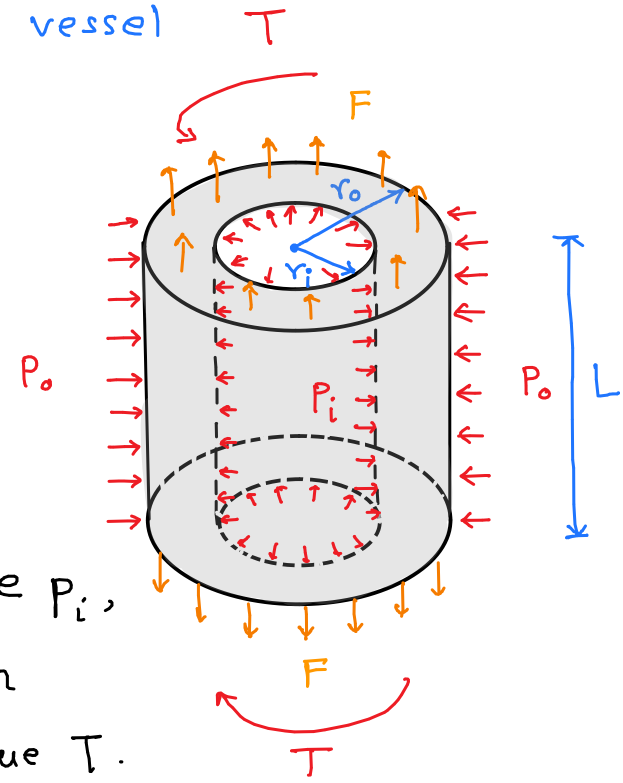
(2) Solutions of long slender bar subjected to transverse load

These two particular applications allow for simplifications (or approximations) to be made to the full 3D equations of elasticity, particularly the linear elastic stress-strain relations. This will allow us to write down simple expression for the stress and strain and solve some important practical problems analytically.

Elastic Solution of Thick-walled cylinders

Here we consider an application of theory of elasticity for homogeneous isotropic solids which are in the form of cylindrical objects called pressure vessel

A typical pressure vessel is a cylinder with inner radius r_i and outer radius r_o as shown.



The pressure vessel may be subjected to uniform inner pressure P_i, uniform outer pressure P_o, uniform axial tensile force F, and torque T.

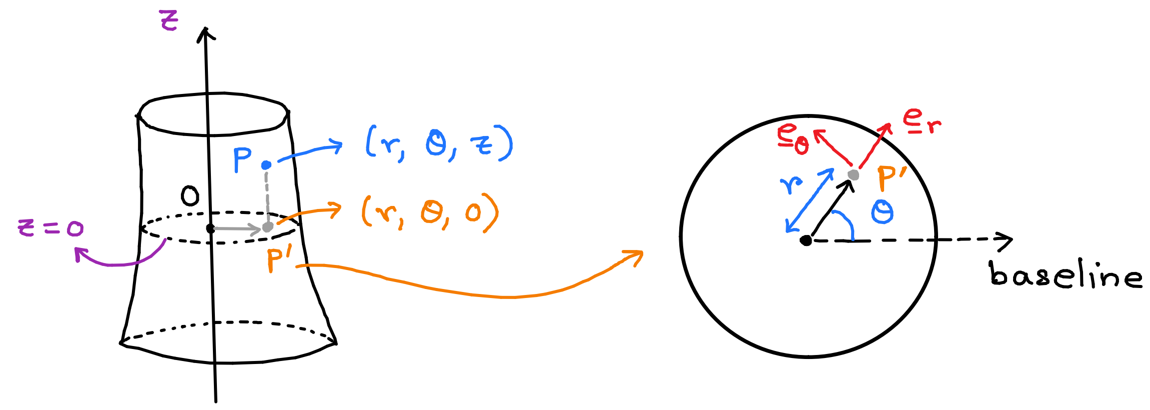
For example, for the case of a submersible hull like that of Oceangate, the important load is the outer pressure P_o

If the length L were small compared with the radii, the cylinder will resemble a disc and the important load might the inner pressure which arises from a "shrink-fit" attachment to a shaft.

We shall determine the complete solution: distribution of internal stresses, strains, and displacements using the 15 eqns : equilibrium eqns, strain-displacement relations and homogeneous isotropic linear elastic stress-strain relations and BCs.

To take advantage of the cylindrical symmetry, we use the cylindrical coordinates (r, θ, z)

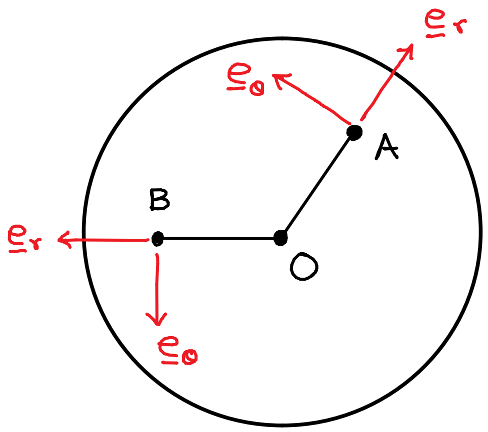
Short introduction to cylindrical coordinates



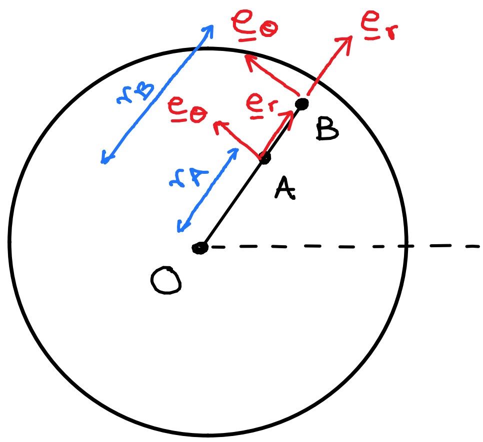
A generalized cylinder with its axis coinciding with z-axis

Position vector of any point P will be given by $r\,\underline{e}_r + z\,\underline{e}_z$ where the θ dependence is hidden in the basis vector $\underline{e}_r$. If we project the point P on $z=0$ plane, we get P' which is given by $(r, \theta, 0)$. There is a baseline or a reference line relative to which angle θ is measured. Two of the basis vectors lie in the plane: $\underline{e}_r$ points radially outward from the center while $\underline{e}_\theta$ points in the direction of increasing θ and is perpendicular to $\underline{e}_r$. The third basis vector $\underline{e}_z$ lies along the axis of the cylinder.

Note: A big difference between cylindrical CS $(\underline{e}_r, \underline{e}_\theta, \underline{e}_z)$ and Cartesian CS $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ is that Cartesian CS is fixed in direction and do not change from one point to the other point, but in cylindrical CS, two of the basis vectors $(\underline{e}_r, \underline{e}_\theta)$ change when θ coordinate of a pt changes



Two points A and B with different θ have different orientations of e_r and e_θ



Two points A and B with same orientation θ has the same directions of e_r and e_θ

Partial derivatives of e_r and e_θ w.r.t. θ

Since the directions of e_r and e_θ depend upon the orientation θ, the following relation is obtained:

$$\frac{\partial e_r}{\partial \theta} = e_\theta \qquad \text{and} \qquad \frac{\partial e_\theta}{\partial \theta} = -e_r$$

Equilibrium equations in cylindrical coordinates

We had in Tutorial 3 (Problem 8) derived the equilibrium equations in cylindrical coordinates, rewritten here:

r-direction

body force

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + b_r = 0$$

θ-direction

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2\tau_{r\theta}}{r} + b_\theta = 0$$

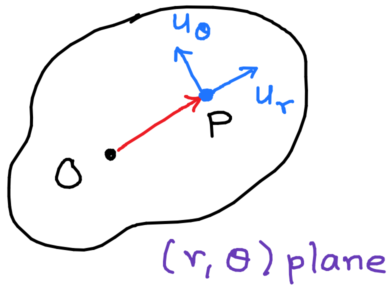
z-direction

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\tau_{zr}}{r} + b_z = 0$$

The 15 equations of elasticity in cylindrical coordinates consist of the 3 equilibrium conditions (on previous page), 6 strain-displacement relations (which we will derive) and 6 stress-strain relations in cylindrical coordinates

Strain-displacement relations in cylindrical coordinates

The displacement vector in the cylindrical coordinate is written as



$$\underline{u} = u_r \underline{e}_r + u_\theta \underline{e}_\theta + u_z \underline{e}_z$$

The displacement of the point P

- in the radial direction $\rightarrow u_r$
- in the θ - direction $\rightarrow u_\theta$
- in the z - direction $\rightarrow u_z$

Normal strain in the r-direction

$$\epsilon_{rr} = \underline{e}_r \cdot \frac{\partial \underline{u}}{\partial r} = \underline{e}_r \cdot \left(\frac{\partial u_r}{\partial r} \underline{e}_r + \frac{\partial u_\theta}{\partial r} \underline{e}_\theta + \frac{\partial u_z}{\partial r} \underline{e}_z \right)$$

$$= \frac{\partial u_r}{\partial r}$$

Normal strain in the z - direction

$$\epsilon_{zz} = \underline{e}_z \cdot \frac{\partial \underline{u}}{\partial z} = \underline{e}_z \cdot \left(\frac{\partial u_r}{\partial z} \underline{e}_r + \frac{\partial u_\theta}{\partial z} \underline{e}_\theta + \frac{\partial u_z}{\partial z} \underline{e}_z \right) = \frac{\partial u_z}{\partial z}$$

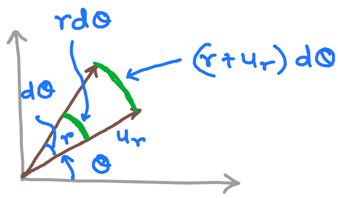
Normal strain in the θ-direction, however turns out to be

$$\epsilon_{\theta\theta} = \underline{e}_\theta \cdot \frac{1}{r}\frac{\partial \underline{u}}{\partial \theta}$$

$$= \underline{e}_\theta \cdot \frac{1}{r}\Bigg(\underbrace{\frac{\partial u_r}{\partial \theta}\underline{e}_r + u_r \overset{\underline{e}_\theta}{\frac{\partial \underline{e}_r}{\partial \theta}}}_{\frac{\partial}{\partial \theta}(u_r \underline{e}_r)} + \underbrace{\frac{\partial u_\theta}{\partial \theta}\underline{e}_\theta + u_\theta \overset{-\underline{e}_r}{\frac{\partial \underline{e}_\theta}{\partial \theta}}}_{\frac{\partial}{\partial \theta}(u_\theta \underline{e}_\theta)} + \frac{\partial u_z}{\partial \theta}\underline{e}_z\Bigg)$$

$$= \frac{1}{r}\frac{\partial u_{\theta\theta}}{\partial \theta} + \underbrace{\frac{u_r}{r}}_{}$$

additional term coming from radial displacement



$$(e_{\theta\theta})_{\text{due to } u_r} = \frac{(r+u_r)\,d\theta - r\,d\theta}{r\,d\theta} = \frac{u_r}{r}$$

Shear strain in r-θ direction

$$\gamma_{r\theta} = \underline{e}_r \cdot \frac{1}{r}\frac{\partial \underline{u}}{\partial \theta} + \underline{e}_\theta \cdot \frac{\partial \underline{u}}{\partial r}$$

$$= \underline{e}_r \cdot \frac{1}{r}\Bigg(\frac{\partial u_r}{\partial \theta}\underline{e}_r + u_r \overset{\underline{e}_\theta}{\frac{\partial \underline{e}_r}{\partial \theta}} + \frac{\partial u_\theta}{\partial \theta}\underline{e}_\theta + u_\theta \overset{-\underline{e}_r}{\frac{\partial \underline{e}_\theta}{\partial \theta}} + \frac{\partial u_z}{\partial \theta}\underline{e}_z\Bigg)$$

$$+ \underline{e}_\theta \cdot \left(\frac{\partial u_r}{\partial r}\underline{e}_r + \frac{\partial u_\theta}{\partial r}\underline{e}_\theta + \frac{\partial u_z}{\partial r}\underline{e}_z\right)$$

$$= \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}$$

Similarly, shear strain in θ-z direction

$$\gamma_{\theta z} = \frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}$$

shear strain in r-z direction

$$\gamma_{rz} = \frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}$$

In summary, the 6 strain-displacement relations in cylindrical coordinates are :

$$\epsilon_{rr} = \frac{\partial u_r}{\partial r}$$

$$\epsilon_{zz} = \frac{\partial u_z}{\partial z}$$

$$\epsilon_{\theta\theta} = \frac{1}{r}\frac{\partial u_{\theta\theta}}{\partial \theta} + \frac{u_r}{r}$$

$$\gamma_{r\theta} = \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}$$

$$\gamma_{\theta z} = \frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}$$

$$\gamma_{rz} = \frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}$$

Stress-strain relations in cylindrical coordinates

For an isotropic material, all stress-strain relations are independent of the direction. Thus, the $\sigma-\epsilon$ relationship must also be independent of the coordinate system, meaning one could choose any set of three perpendicular directions. Therefore, we can write

$$\epsilon_{rr} = \frac{1}{E}\left(\sigma_{rr} - \nu(\sigma_{\theta\theta} + \sigma_{zz})\right)$$

$$\epsilon_{\theta\theta} = \frac{1}{E}\left(\sigma_{\theta\theta} - \nu(\sigma_{rr} + \sigma_{zz})\right)$$

$$\epsilon_{zz} = \frac{1}{E}\left(\sigma_{zz} - \nu(\sigma_{rr} + \sigma_{\theta\theta})\right)$$

$$\gamma_{r\theta} = \tau_{r\theta} / G$$

$$\gamma_{rz} = \tau_{rz} / G$$

$$\gamma_{\theta z} = \tau_{\theta z} / G$$

OR

$$\sigma_{rr} = \lambda(\epsilon_{rr} + \epsilon_{\theta\theta} + \epsilon_{zz}) + 2\mu\,\epsilon_{rr}$$

$$\sigma_{\theta\theta} = \lambda(\epsilon_{rr} + \epsilon_{\theta\theta} + \epsilon_{zz}) + 2\mu\,\epsilon_{\theta\theta}$$

$$\sigma_{zz} = \lambda(\epsilon_{rr} + \epsilon_{\theta\theta} + \epsilon_{zz}) + 2\mu\,\epsilon_{zz}$$

$$\tau_{r\theta} = \mu\,\gamma_{r\theta}$$

$$\tau_{rz} = \mu\,\gamma_{rz}$$

$$\tau_{\theta z} = \mu\,\gamma_{\theta z}$$