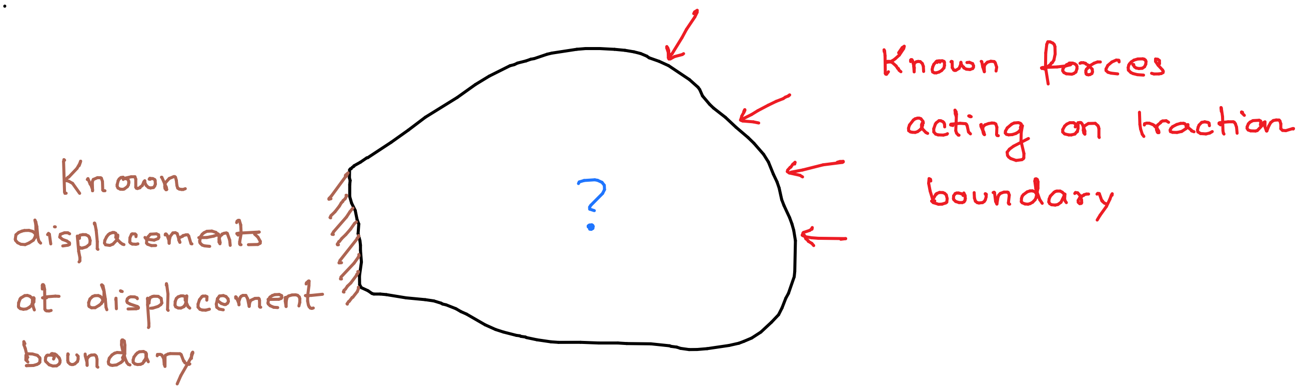
Mechanics Modelling

In problems of mechanics, one usually knows (some of) the forces (or stresses) acting on the body under study, be it due to wind, water pressure, the weight of the human body, a moving train, and so on → Traction BC

One also often knows something about the displacements along some portion of the body, for example, it might be fixed to the ground and so the displacements there are zero. → Displacement BC

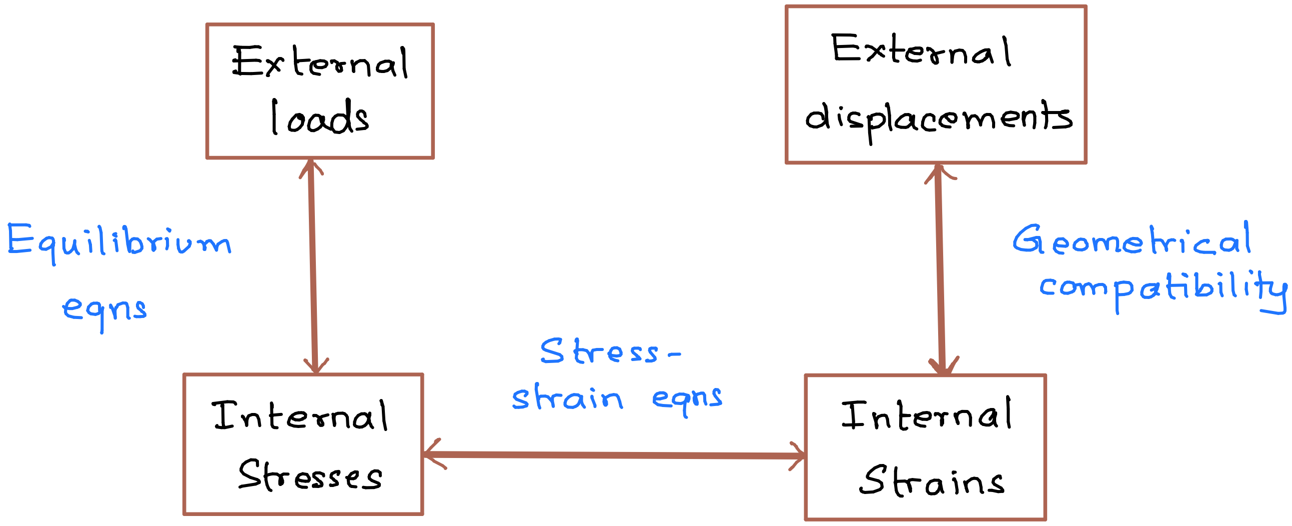


The basic problem of mechanics is to determine what is happening INSIDE the body.

This means: what are the stresses and strains inside?

- Where are the stresses high?
- Where will the material first fail?
- What can we change to make the material function better?

The problems that we solve using theory of mechanics is to find distributions of stresses and strains which satisfy the prescribed loads and displacements on the boundary and which satisfy the equilibrium conditions, the stress-strain-temp relations, and the geometrical strain-displacement relations



Complete equations of linear elasticity

As previously mentioned that unlike rigid bodies, deformable solid bodies require satisfying geometric compatibility and stress-strain-temperature relations in addition to equilibrium eqns

Equilibrium equations (3 eqns)

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \tau_{12}}{\partial x_2} + \frac{\partial \tau_{13}}{\partial x_3} + \gamma_1 = 0$$

body forces (force per unit volume)

$$\frac{\partial \tau_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \tau_{23}}{\partial x_3} + \gamma_2 = 0$$

$$\frac{\partial \tau_{13}}{\partial x_1} + \frac{\partial \tau_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + \gamma_3 = 0$$

Geometric Compatibility / Strain displacement relations (6 eqns)

The displacements must match the geometrical boundary condns and must be continuous functions of position with which the strain components are associated

$$\epsilon_{11} = \frac{\partial u_1}{\partial x_1}, \qquad \gamma_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}$$

$$\epsilon_{22} = \frac{\partial u_2}{\partial x_2}, \qquad \gamma_{23} = \frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}$$

$$\epsilon_{33} = \frac{\partial u_3}{\partial x_3}, \qquad \gamma_{13} = \frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}$$

Stress-strain-temperature relations (6 eqns)

$$\epsilon_{11} = \frac{1}{E}\left[\sigma_{11} - \nu(\sigma_{22} + \sigma_{33})\right] + \alpha\,(T - T_0)$$

$$\epsilon_{22} = \frac{1}{E}\left[\sigma_{22} - \nu(\sigma_{11} + \sigma_{33})\right] + \alpha\,(T - T_0)$$

$$\epsilon_{33} = \frac{1}{E}\left[\sigma_{33} - \nu(\sigma_{11} + \sigma_{22})\right] + \alpha\,(T - T_0)$$

$$\gamma_{12} = \frac{\tau_{12}}{G}, \qquad \gamma_{23} = \frac{\tau_{23}}{G}, \qquad \gamma_{13} = \frac{\tau_{13}}{G}$$

The equilibrium eqns (3 eqns) + strain-displacement (6 eqns) + stress-strain-temperature (6 eqns) provide 15 equations

Alongside there are : 15 unknowns

- 6 stresses $(\sigma_{11}, \sigma_{22}, \sigma_{33}, \tau_{12}, \tau_{13}, \tau_{23})$
- 6 strains $(\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, \gamma_{12}, \gamma_{13}, \gamma_{23})$
- 3 displacements (u_1, u_2, u_3)

These 15 equations are the fundamental equations of linear elasticity theory

Note the equations are LINEAR ELASTIC due to:

(a) Assumption of linear material behavior (stress – strain)

$$\underbrace{\sigma = f(\epsilon)}_{\text{Usually nonlinear}} \implies \text{Linear assumption} \Rightarrow \sigma \propto \epsilon$$

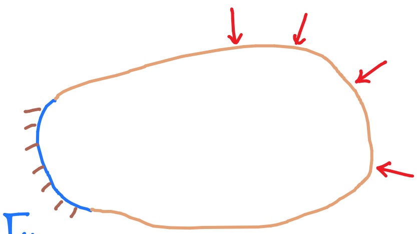
(b) Restriction to small strains leading to linear strain – displacement relation; small strains usually lead to linear stress-strain relation

(c) the material returns to its original shape when the loads are removed, and the unloading path is same as the loading path (elastic)

(d) there is no dependence on the strain rate (elastic)

Use of boundary conditions for solution

In order to obtain solutions to the 15 equations, it is necessary to perform integrations because of the derivatives that appear in equilibrium and strain-displacement equations. To fix the solution for a particular body, it is necessary to prescribe boundary conditions at every point on the surface of the body. Every point on the surface of the body will have either the displacement vector or the surface traction vector specified.



$$\Gamma = (\Gamma_u \cup \Gamma_t)$$

The applied loads are usually characterized by specifying traction vector acting on a portion, Γ_t, of the total surface Γ of the body.

$$\boxed{\underline{T}^{(n)} = \underline{T}^* \quad \text{on} \quad \Gamma_t}$$

Traction BC

known applied loading

Moreover, supports are specified over the portion of the surface Γ_u, where the displacement vector $\underline{u}$ is specified.

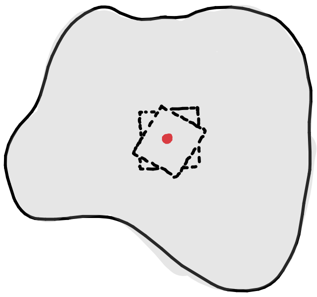
$$\boxed{\underline{u} = \underline{u}^* \quad \text{on} \quad \Gamma_u}$$

Displacement BC

known support conditions

Other types of materials

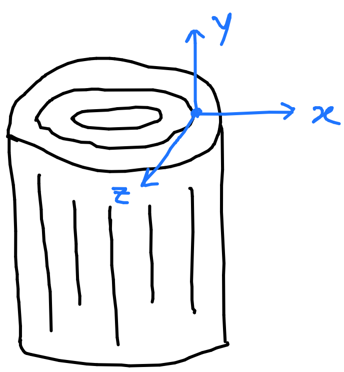
A material whose properties are the same in all directions is called isotropic. In particular, the relationship between stress and strain at any single point in the body is same in all directions. This implies that if a specimen is cut from an isotropic material and subjected to a load, it would not matter in which orientation the specimen is cut, the resulting deformation would be the same.



Stress-strain relation remains invariant to rotation

There are materials which are not isotropic but commonly seen in nature. The classic example is wood, hich has a clear directional depend nce — structure along the fibers (lines that can be seen) are different from those perpendicular to the fibers. As such, wood is stiffer and stronger along the fiber direction than perpendicular to them.

Wood is an orthotropic material → stress-strain relations are independent in three mutually perpendicular directions.



(has 9 independent elastic constants)

Anisotropic materials are materials which show different stress-strain behavior in different directions. Wood is also sometimes modelled as anisotropic material. Anisotropic linear elastic materials require 21 independent elastic constants.

Homogeneous materials

For materials where the stress-strain relations are same at each point throughout are called homogeneous. In other words, the relationship between stress and strain is the same for all material particles. Most materials are assumed to be homogeneous.

Note that isotropy is related to rotational invariance of stress-strain relation at a point, whereas homogeneity is related to translational invariance of stress-strain relations. Therefore, a material could be homogeneous but not isotropic (for ex., wood). Remember that homogeneous refers to different locations whereas isotropy refers to same location but different directions.