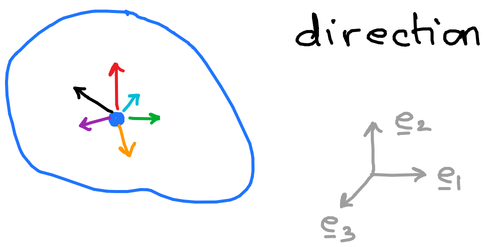
Stress-strain relations for isotropic linear elastic materials

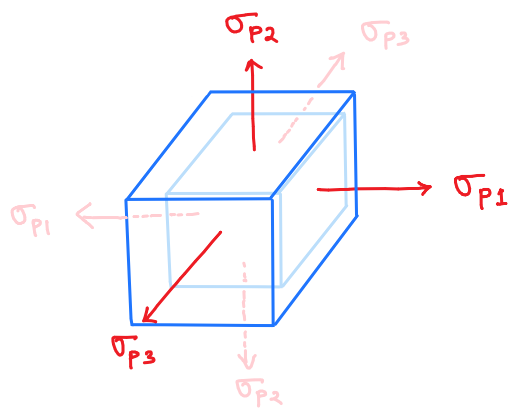
We had obtained that for an anisotropic linear elastic material there are 21 elastic constants

ANISOTROPIC $\rightarrow$ At a point in the body, the material properties are different in different directions



ISOTROPIC $\rightarrow$ Material properties are same in all directions

Lets consider a body made of isotropic material in a coordinate system defined by principal directions $(\underline{n}_1 - \underline{n}_2 - n_3)$



When a small cuboidal element is subjected to principal stresses on its faces, the element will remain cuboidal after deformation (no distortion / shape change).

Thus, the normals to these faces must coincide with the principal strain directions.

Hence, for an isotropic material, one can relate the principal stresses σ_{p1}, σ_{p2}, σ_{p3} with three principal strains ϵ_{p1}, ϵ_{p2}, ϵ_{p3} through suitable elastic constants.

For principal direction $\underline{n}_1$:

$$\sigma_{p1} = C_{11}\,\epsilon_{p1} + C_{12}\,\epsilon_{p2} + C_{13}\,\epsilon_{p3}$$

We note that C_{12} and C_{13} must be equal since the effect of σ_{p1} in the directions of ϵ_{p2} and ϵ_{p3} (i.e. $\underline{n}_2$ and $\underline{n}_3$), which are perpendicular to $\underline{n}_1$, must be the same for an isotropic material. Hence, for σ_{p1}, the equation is

$$\begin{aligned}\sigma_{p1} &= C_{11}\,\epsilon_{p1} + C_{12}\,(\epsilon_{p2} + \epsilon_{p3}) \\ &= \underbrace{(C_{11} - C_{12})}_{2\mu}\,\epsilon_{p1} + \underbrace{C_{12}}_{\lambda}\,\underbrace{(\epsilon_{p1} + \epsilon_{p2} + \epsilon_{p3})}_{\substack{\text{1st invariant of strain } J_1 \\ \text{(also volumetric strain } \epsilon_v)}}\end{aligned}$$

$$\sigma_{p1} = \lambda\,\mathrm{tr}(\underline{\underline{\epsilon}}) + 2\mu\,\epsilon_{p1}$$

By similar arguments, we can show that

$$\sigma_{p2} = \lambda\,\mathrm{tr}(\underline{\underline{\epsilon}}) + 2\mu\,\epsilon_{p2}$$

$$\sigma_{p3} = \lambda\,\mathrm{tr}(\underline{\underline{\epsilon}}) + 2\mu\,\epsilon_{p3}$$

$\boxed{\lambda, \mu \rightarrow \text{LAME'S CONSTANTS}}$

For a general coordinate system, it can be shown that the six stress components $\sigma_{11}, \sigma_{22}, \sigma_{33}, \tau_{12}, \tau_{13}, \tau_{23}$ are related to six strain components $\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, \epsilon_{12}, \epsilon_{13}, \epsilon_{23}$

$$\sigma_{11} = \lambda \, tr(\underline{\underline{\epsilon}}) + 2\mu \, \epsilon_{11}$$

$$\sigma_{22} = \lambda \, tr(\underline{\underline{\epsilon}}) + 2\mu \, \epsilon_{22}$$

$$\sigma_{33} = \lambda \, tr(\underline{\underline{\epsilon}}) + 2\mu \, \epsilon_{33}$$

$\lambda, \mu \rightarrow$ Lame's constants

$$\tau_{12} = 2\mu \, \epsilon_{12}$$

$$\tau_{13} = 2\mu \, \epsilon_{13}$$

$$\tau_{23} = 2\mu \, \epsilon_{23}$$

2 elastic constants for Isotropic Linear Elastic material

Alternatively, we can use another form where strain is expressed in terms of stress. It is also called three-dimensional Hooke's law of Elasticity, and can be written as follows:

$$\epsilon_{11} = \frac{1}{E}\left(\sigma_{11} - \nu(\sigma_{22} + \sigma_{33})\right)$$

$$\epsilon_{22} = \frac{1}{E}\left(\sigma_{22} - \nu(\sigma_{11} + \sigma_{33})\right)$$

$$\epsilon_{33} = \frac{1}{E}\left(\sigma_{33} - \nu(\sigma_{11} + \sigma_{22})\right)$$

In this representation, we have three constants:

E — Young's Modulus

ν — Poisson's Ratio

G — Shear Modulus

$$\gamma_{12} = 2\epsilon_{12} = \frac{\tau_{12}}{G}$$

$$\gamma_{13} = 2\epsilon_{13} = \frac{\tau_{13}}{G}$$

$$\gamma_{23} = 2\epsilon_{23} = \frac{\tau_{23}}{G}$$

However, there are only two indep. constants. Therefore, we have

$$\boxed{G = \frac{E}{2(1+\nu)}}$$

Physical Significance of E, G and ν

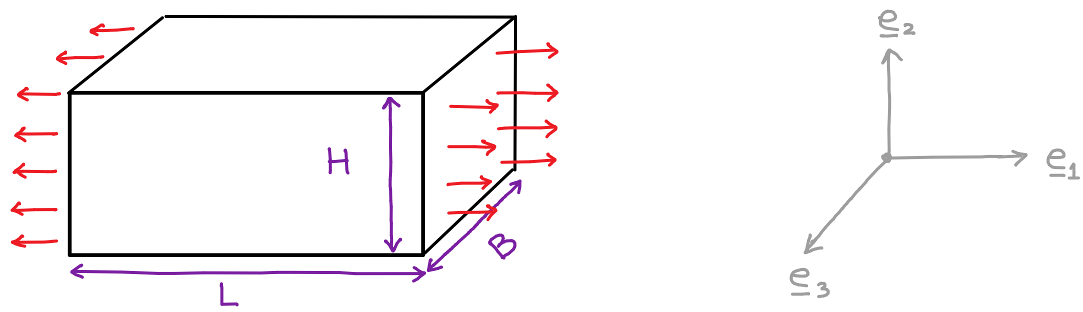
$$\epsilon_{11} = \frac{1}{E}\left(\sigma_{11} - \nu(\sigma_{22}+\sigma_{33})\right)$$

$$\epsilon_{22} = \frac{1}{E}\left(\sigma_{22} - \nu(\sigma_{11}+\sigma_{33})\right)$$

$$\epsilon_{33} = \frac{1}{E}\left(\sigma_{33} - \nu(\sigma_{11}+\sigma_{22})\right)$$

We can see that the normal strain in one direction not only depends on the normal stress in that direction but also on the normal stresses in other two directions.

To understand the physical relevance, we can think of a tensile testing. Suppose that we have a rectangular beam of length L, breadth B and height H.

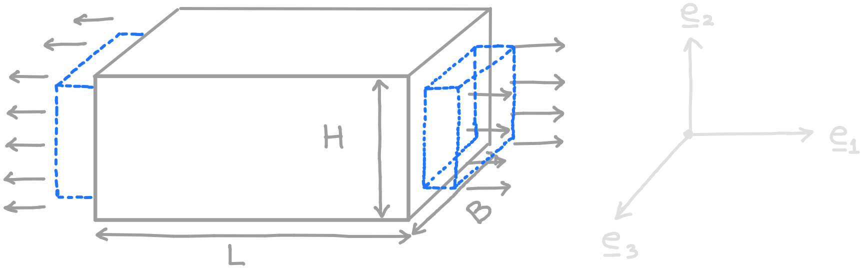


The beam is oriented such that its length is along $\underline{e}_1$, its height is along $\underline{e}_2$ and its breadth is along $\underline{e}_3$.

Let's apply force on the left and right faces to stretch the beam. So σ_{11} will be non-zero while the shear components τ_{12} and τ_{13} will be zero. Also, we are not applying any force on the lateral surface ($\underline{e}_2$ and $\underline{e}_3$ planes), so there is no stress on them. Infact, any internal section with normal along $\underline{e}_2$ & $\underline{e}_3$ will not have any traction component.

The state of stress in this case will then be

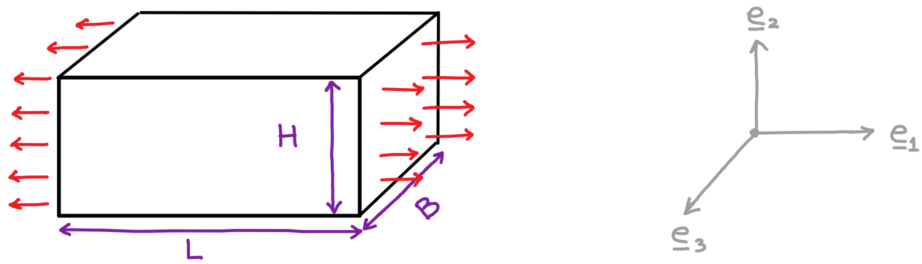
$$\begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



This stress will lead to some strain in the body. We know that ϵ_{11} is generated because the length of the beam changes. However, ϵ_{22} and ϵ_{33} are also generated: due to stretching in one direction, there will be contraction in other two directions. But there will be no shear strain generated if we are careful in stretching the body uniformly. Thus, the state of strain for the rectangular beam will be

$$\begin{bmatrix} \epsilon_{11} & 0 & 0 \\ 0 & \epsilon_{22} & 0 \\ 0 & 0 & \epsilon_{33} \end{bmatrix}$$

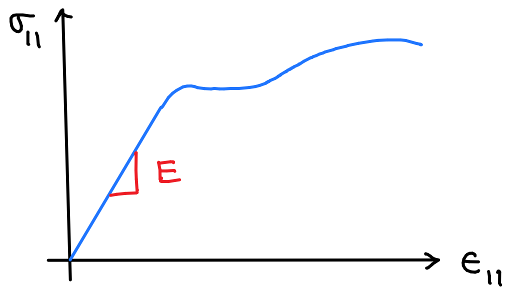
(1) Young's Modulus (E)

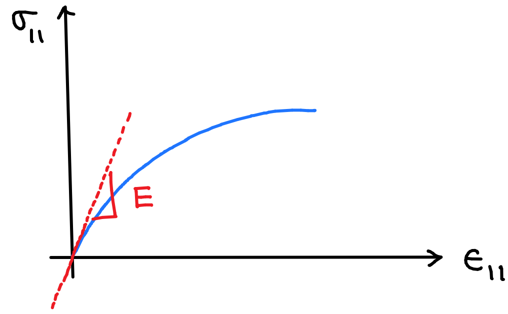


If the elongation is uniform along the length of the bar, the local strain will be equal to the average normal strain. Thus,

$$\epsilon_{11} = \frac{\Delta L}{L}, \qquad \epsilon_{22} = \frac{\Delta H}{H}, \qquad \epsilon_{33} = \frac{\Delta B}{B}$$

If we now draw the stress-strain curve of $\sigma_{11} - \epsilon_{11}$ from a tensile test experiment by measuring the change in length, we may get a curve



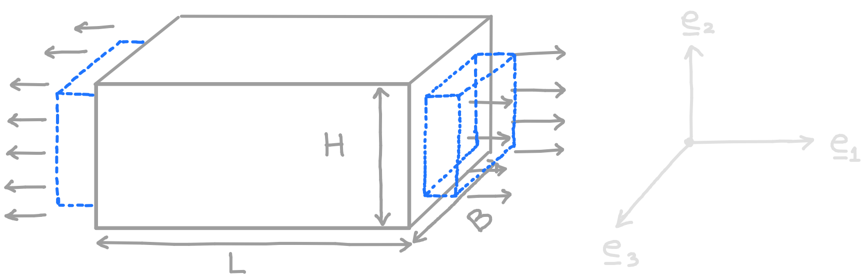


The initial slope of this graph gives us the Young's modulus

$$E = \left.\frac{\partial \sigma_{11}}{\partial \epsilon_{11}}\right|_{\epsilon_{11}=0}$$

While computing this derivative from the graph, we should not have σ_{22} or σ_{33} present. Essentially, the rectangular beam should be stretched in such a way that it can freely shrink in the lateral directions.

For a uniaxial tensile test experiment, we can say that



$$\sigma_{22} = \sigma_{33} = 0$$

$$\sigma_{11} = E\,\epsilon_{11}$$

(for small ϵ_{11})

(2) Poisson's ratio (ν)

The Poisson's ratio of a material is defined as

$$\nu = - \frac{\text{Lateral normal strain}}{\text{Longitudinal normal strain}}$$

In a uniaxial tensile test, we directly impose longitudinal normal strain in the e_1-direction and this, in turn, induces strain along e_2 and e_3 directions. For an isotropic body, the lateral strains in these directions will be equal. Thus, the Poisson's ratio from a uniaxial tensile experiment would be

$$\nu = - \frac{\epsilon_{22}}{\epsilon_{11}} = - \frac{\epsilon_{33}}{\epsilon_{11}}$$

We can also derive the Poisson's ratio from the stress-strain relation

In the tensile test,

$$\sigma_{22} = \sigma_{33} = 0$$

- $\epsilon_{11} = \frac{1}{E} \left(\sigma_{11} - \nu \left(\cancelto{0}{\sigma_{22}} + \cancelto{0}{\sigma_{33}} \right) \right)$

  $\Rightarrow \quad \epsilon_{11} = \sigma_{11}/E$

- $\epsilon_{22} = \frac{1}{E} \left(\cancelto{0}{\sigma_{22}} - \nu \left(\sigma_{11} + \cancelto{0}{\sigma_{33}} \right) \right)$

  $\Rightarrow \quad \epsilon_{22} = -\frac{\nu}{E} \sigma_{11} = -\nu \epsilon_{11}$

- $\epsilon_{33} = \frac{1}{E} \left(\cancelto{0}{\sigma_{33}} - \nu \left(\sigma_{11} + \cancelto{0}{\sigma_{22}} \right) \right)$

  $\Rightarrow \quad \epsilon_{33} = -\frac{\nu}{E} \sigma_{11} = -\nu \epsilon_{11}$

We get the same formula

(3) Shear Modulus (G)

$$\gamma_{12} = 2\epsilon_{12} = \frac{\tau_{12}}{G}$$

$$\gamma_{13} = 2\epsilon_{13} = \frac{\tau_{13}}{G}$$

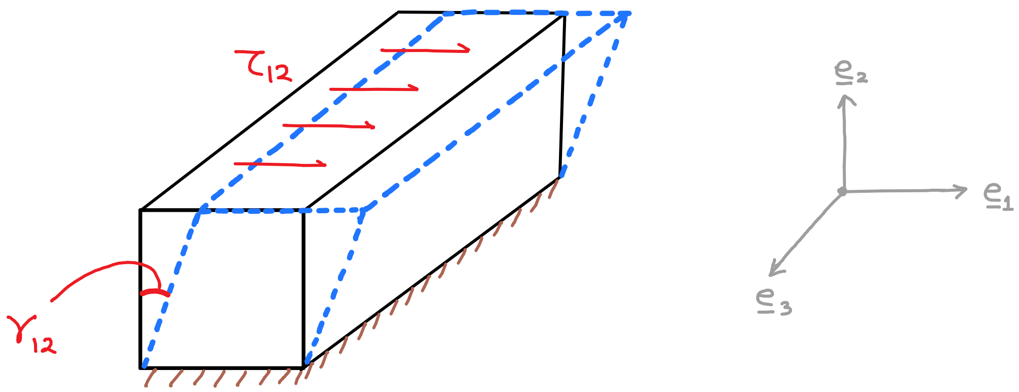
$$\gamma_{23} = 2\epsilon_{23} = \frac{\tau_{23}}{G}$$

We can see that shear modulus is given by

$$G = \frac{\tau_{12}}{\gamma_{12}} = \frac{\tau_{13}}{\gamma_{13}} = \frac{\tau_{23}}{\gamma_{23}}$$

So if we induce shear in a body and measure the corresponding shear stress, the ratio of stress to strain will give us the shear modulus.

Motivated by this, let's do an experiment with the rectangular bar, where the bar is fixed at the bottom and we apply shear force on the top face (with normal $\underline{e}_1$)



This effectively imposes shear stress τ_{12} in the body. Due to this, the bar shears: the initially perpendicular edges of the front face get inclined and shear strain γ_{12} would be equal to $\left(\frac{\pi}{2} - \text{angle between two edges}\right)$. If we draw a τ_{12} vs γ_{12} curve, the initial slope of the curve gives $G = \left.\frac{\tau_{12}}{\gamma_{12}}\right|_{\gamma_{12}=0}$

Bulk Modulus of Elasticity (K)

Young's modulus of elasticity relates normal stress with normal strain. Shear modulus relates shear stress with shear strain. We will now define bulk modulus of elasticity which relates volumetric strain (ϵ_v) with volumetric stress or equivalent pressure. When we apply pressure to a fluid (liquid/gas), its volume decreases. This decrease can be quantified by volumetric strain given by

$$\text{Volumetric strain, } \epsilon_v = \frac{\Delta V}{V}$$

If a pressure increase of ΔP generates volumetric strain in a fluid, bulk modulus is then given by

$$K_{fluid} = - \frac{\Delta P}{\Delta V / V}$$

For solids, the equivalent pressure is obtained from the hydrostatic part of the stress tensor, P_{eq}

$$P_{eq} = - \frac{I_1(\underline{\sigma})}{3}$$

The negative sign comes because pressure is compressive in nature while the normal component of traction is tensile when positive.

Therefore,

$$K_{solids} = - \frac{P_{eq}}{\epsilon_v} = - \left(\frac{-I_1(\underline{\sigma})/3}{\epsilon_v} \right)$$

That is,

$$K_{solids} = -\left(\frac{-I_1(\underline{\underline{\sigma}})/3}{\epsilon_v}\right) = \frac{I_1(\underline{\underline{\sigma}})/3}{tr(\underline{\underline{\epsilon}})} = \frac{1}{3}\frac{tr(\underline{\underline{\sigma}})}{tr(\underline{\underline{\epsilon}})}$$

Bulk modulus of elasticity

The bulk modulus of elasticity can also be expressed in terms of elastic constants E and ν, for isotropic material.

We have the following:

$$\epsilon_{11} + \epsilon_{22} + \epsilon_{33} = \frac{1}{E}(\sigma_{11} + \sigma_{22} + \sigma_{33}) - \frac{2\nu}{E}(\epsilon_{11} + \epsilon_{22} + \epsilon_{33})$$

$$\Rightarrow \quad tr(\underline{\underline{\epsilon}}) = \frac{1-2\nu}{E}\, tr(\underline{\underline{\sigma}})$$

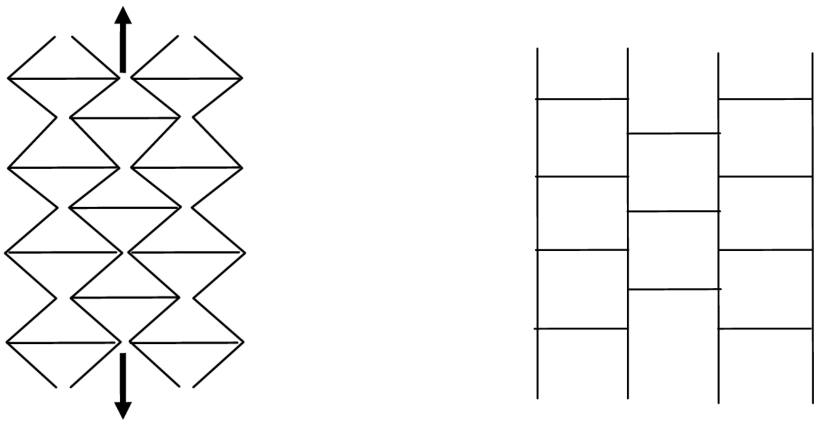
$$\Rightarrow \quad \boxed{K_{solids} = \frac{E}{3(1-2\nu)}}$$

This is an important relation. It tells us that bulk modulus K is not an independent constant. If you know the Young's modulus E and the Poisson's ratio ν, you can get the Bulk modulus using the above relation. Furthermore, this relation also gives an upper limit for the Poisson's ratio. (discussed next)

Theoretical limits for Poisson's ratio

The Poisson's ratio is usually positive. There is however a lot of research in recent years on developing materials having negative Poisson's ratio. Such materials are called auxetic materials.

Auxetic material



(a) Before loading (b) After loading

From the relation,

$$K_{solids} = \frac{E}{3(1-2\nu)} = -\frac{P_{eq}}{\epsilon_v}$$

we can deduce that when ν is very close to $\frac{1}{2}$, the bulk modulus becomes very large. For such materials, if we apply a finite amount of change in equivalent pressure, the volumetric strain induced in the body would be very small. This signifies **incompressibility**. Thus, $\nu \to \frac{1}{2}$ corresponds to the incompressible limit. At other extreme, materials such as cork can have Poisson's ratio close to zero.

To obtain the lower limit for the Poisson's ratio, we can use the relation

$$G = \frac{E}{(1+\nu)}$$

The Young's modulus E and the shear modulus G are both positive quantities. As a result, the denominator in the RHS must also be positive, i.e.

$$2(1+\nu) > 0 \quad \Rightarrow \quad \nu > -1$$

Thus, the theoretical limit for Poisson's ratio:

$$\boxed{-1 < \nu \leqslant 1/2}$$

Thermal Strain

In the elastic region, the effect of temperature on strain appears in two ways:

(a) by causing modification of the elastic constants. Usually the change for most materials is very small and hence is not considered

(b) by directly producing strain in the absence of stress. The strain due to temperature change in the absence of stress is called thermal strain and is denoted by $\underset{=}{\epsilon}^t$

For an isotropic material, thermal strain produces pure expansion or contraction with no shear-strain components. Also, for temp. changes of one or two hundred degrees Fahrenheit, one can closely describe the actual variation by a linear approximation.

The thermal strains due to a change in temperature from T_0 to T

$$\epsilon_{11}^t = \epsilon_{22}^t = \epsilon_{33}^t = \underbrace{\alpha}_{\text{coefficient of linear expansion}} (T - T_0)$$

$$\gamma_{12}^t = \gamma_{23}^t = \gamma_{13}^t = 0$$

The total strain at a point in an elastic body is the sum of that due to stress and that due to temperature.

$$\underbrace{\underset{=}{\epsilon}}_{\text{total strain}} = \underbrace{\underset{=}{\epsilon}^e}_{\text{elastic strain}} + \underbrace{\underset{=}{\epsilon}^t}_{\text{thermal strain}}$$