STRESS - STRAIN relations

Recap

- The ideas of stress and strain at a point were developed separately
  - → Stress used concepts of equilibrium of forces
  - → Strain used only geometry and physical continuity of displacements
- We have three stress-equilibrium relations for six independent stress components
  - → 3 PDEs
- We have six strain-compatibility relations for six independent strain components
  - → 6 PDEs
- We still need more equations to solve for all stresses and strains
- We shall now look at the nature of the material of the body to get more equations
  - → STRESS - STRAIN relations will depend on the material behavior of the body

How to obtain the stress-strain relations?

- Two approaches of deriving stress-strain relations

1) Do experiments at atomic scale

   - X-ray crystallography
   - Electron microscopy
   - Atomic force microscopy

→ determine atomic structure and arrangement of material

↓

Perform atomistic simulations using methods like Molecular Dynamics (MD) or Densil Functional Theory (DFT) to calcula-e atomic-scale properties

↓

Use statistical mechanics and mathematical models to relate atomic-scale properties to macroscopic properties such as Young's modulus

However, such experimentation is costly and time-taking, therefore, they are used not very often.

How to obtain the stress-strain relations? (Contd...)

2) Do experiments at macroscopic level

- Tensile test
- Compression test
- Bending test
- Shear test

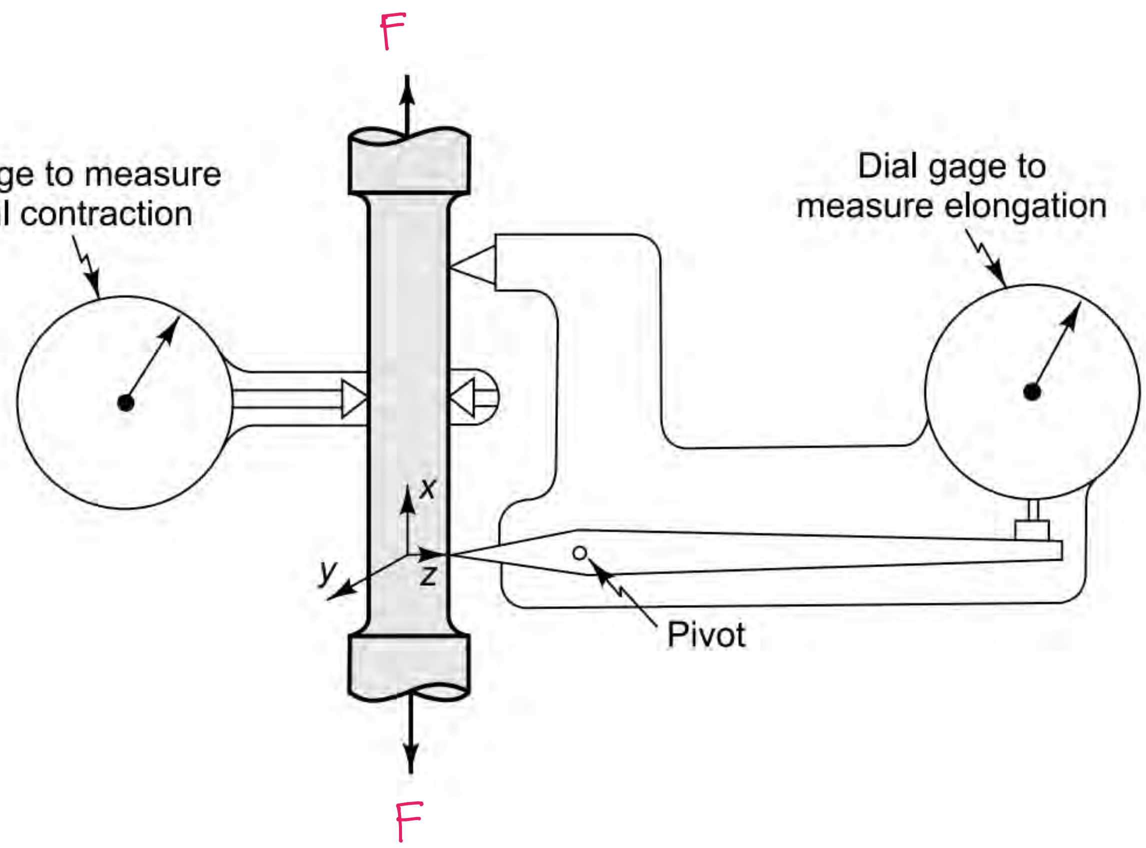
Obtain load-deformation graph

↳ Determine stress-strain curve

↳ Fit mathematical models to the stress-strain curve

↳ Obtain macroscopic properties as parameters of the mathematical models

Tensile test



- A cylindrical dog-bone shaped specimen of constant circular cross-section is pulled in the direction of its axis
- The ends of the specimen are stretched by a testing machine at a slow, constant rate.
- The elongation and lateral contraction are noted as the test proceeds.

This setup represents a UNIAXIAL loading scenario

↓

Only stress component present is σ_{xx} → can be found as

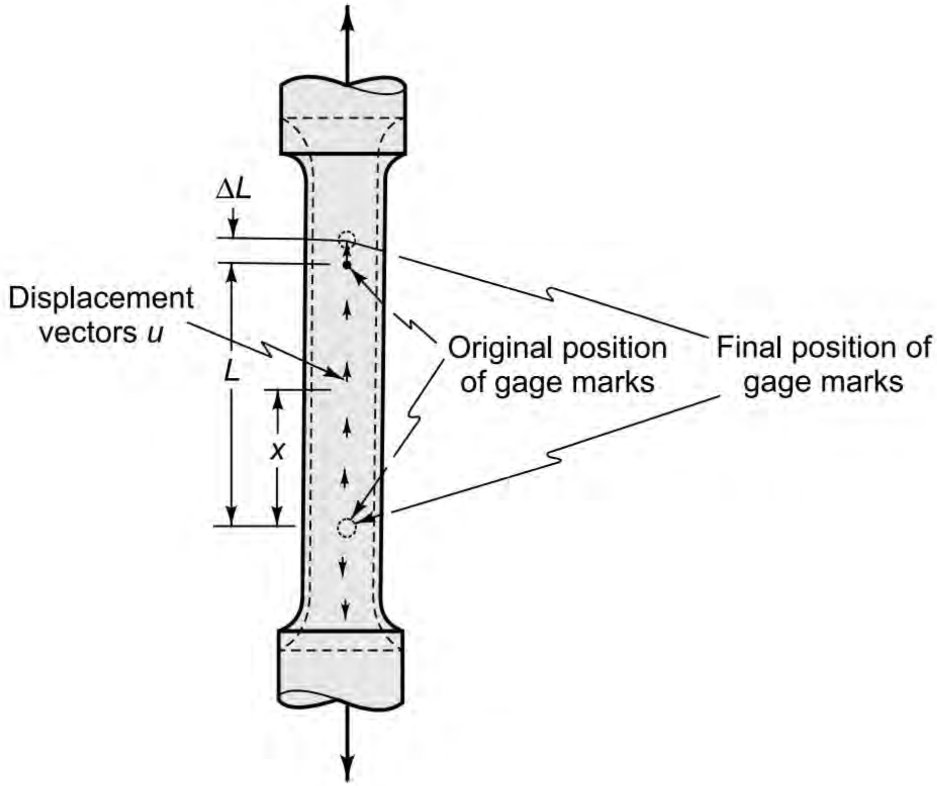
Engineering stress ⟷ $\sigma_{xx} = F/A_0$

Original C/s area

Tensile test

Usually, only axial normal component of strain ϵ_{xx} is reported in a tensile test

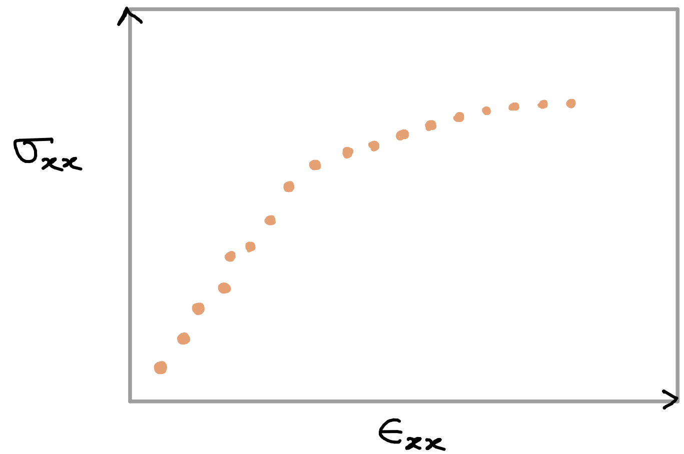
→ To obtain this strain component, the displacement of one point is measured relative to another point at distance L



→ If the displacements vary uniformly over the length L, one may write

$$u_x = \frac{x}{L} \Delta L$$

→ For small strain, $\epsilon_{xx} = \frac{\partial u_x}{\partial x} = \frac{\Delta L}{L}$



Features of stress-strain curves

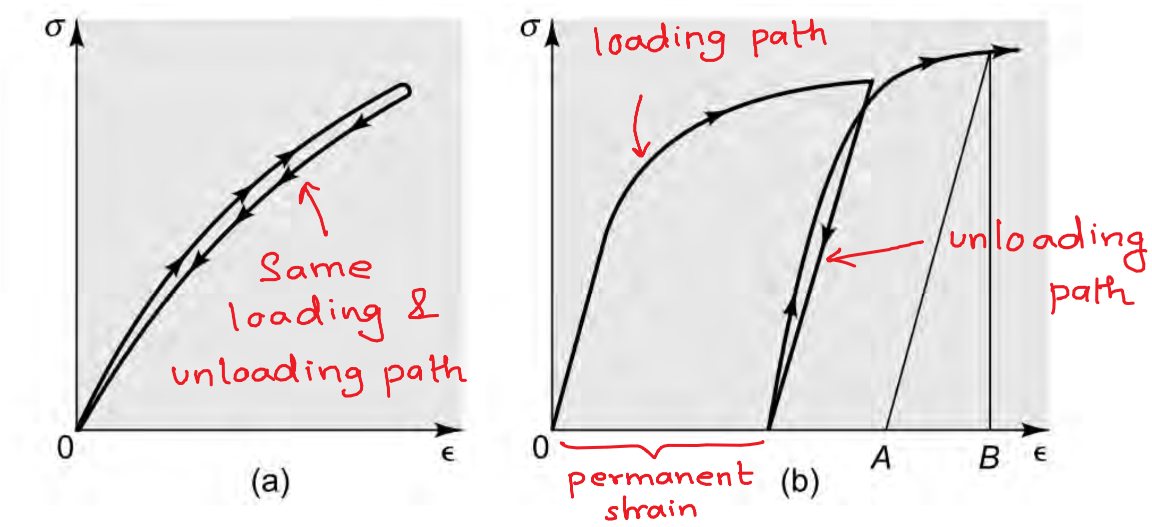
Stress-strain curves of many materials have certain things in common

(a) Elastic region

It is the region where applying stress and then releasing it does not result in any permanent strain

(b) Inelastic region

The region where applying stress and releasing it causes permanent strain in the material



Elastic

Inelastic

Stress-strain curve for ductile materials

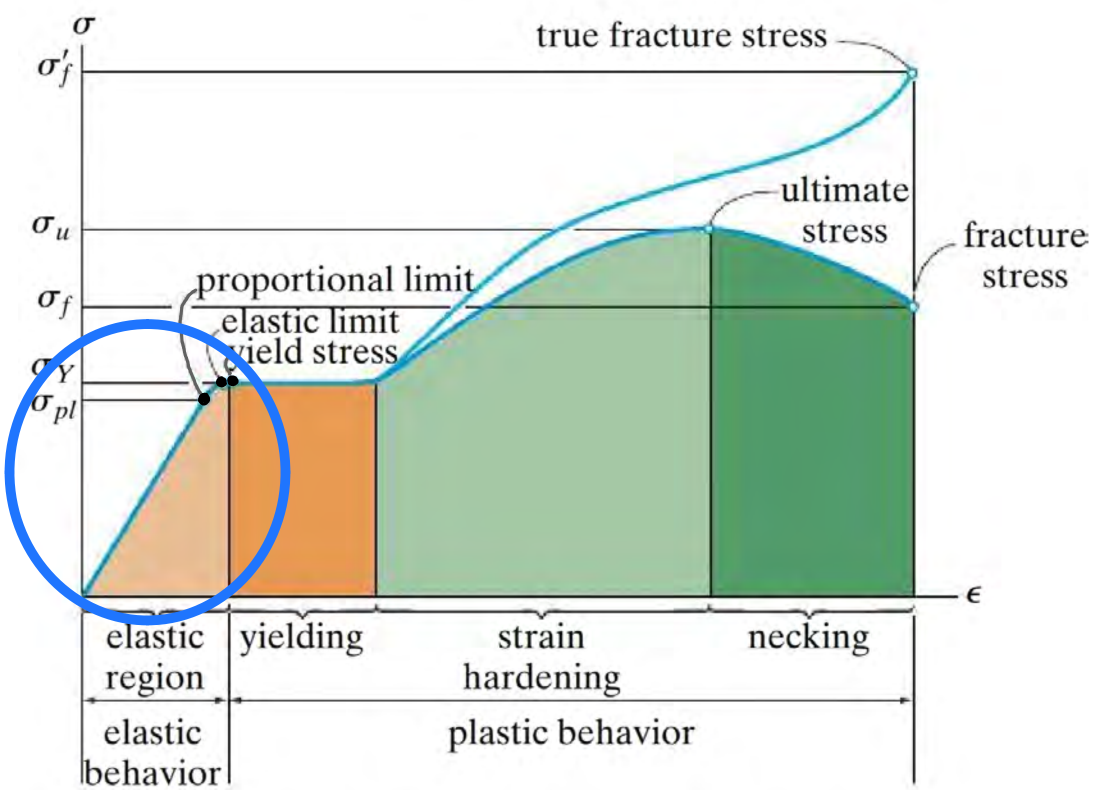
Examples of ductile materials → steel, cast iron, aluminum

- Elastic behavior occurs in the light orange region.

  → The curve is usually a straight line through most of the region. So stress is proportional to strain

  → The material in this region is called linear elastic

  → The upper stress limit to this linear relationship is called proportionality limit, σ_{pl}



Conventional and true stress-strain diagrams for ductile material (steel) (not to scale)

  → If the stress slightly exceeds the proportionality limit, the curve tends to bend and flatten out. This continues until the stress reaches elastic limit.

Difficult to determine in experiments
↓

Stress-strain curve for ductile materials

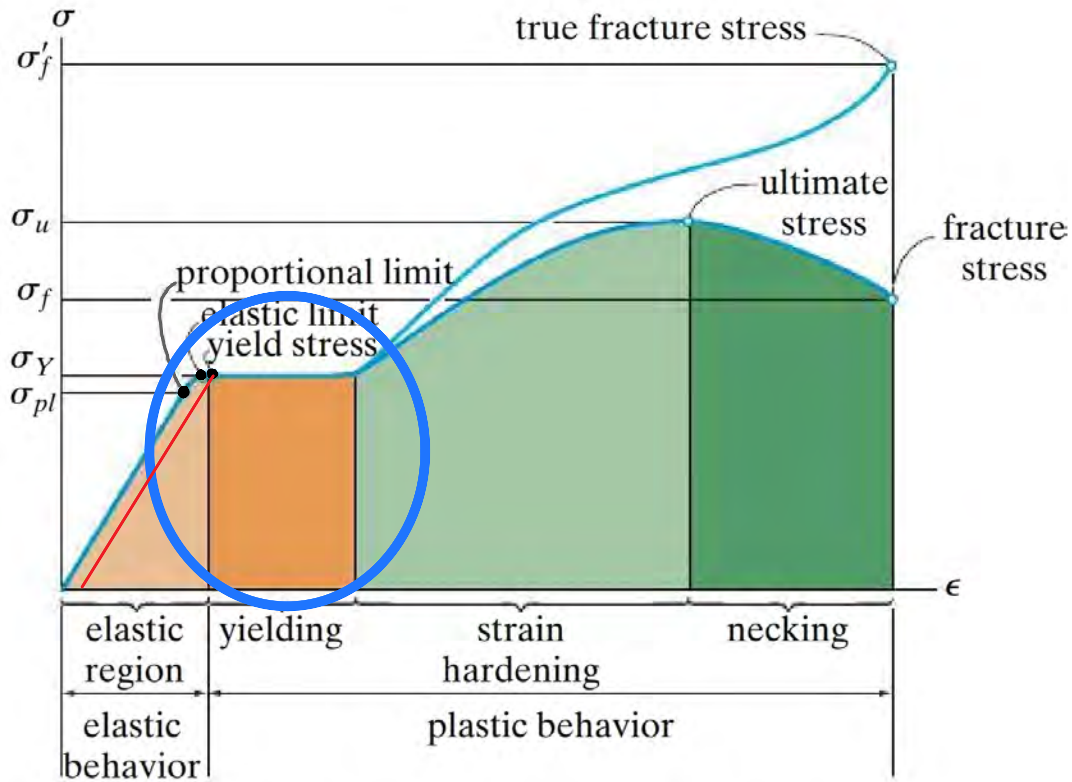
Examples of ductile materials → steel, cast iron, aluminum

- Yielding (Inelastic)

→ A slight increase in stress above the elastic limit will cause it to deform permanently, that is, the body will not fully regain its initial shape

→ Yield stress, σ_y, is the stress at which material continues to deform without further increase in the stress

The associated deformation that occurs is called plastic deformation



Conventional and true stress-strain diagrams for ductile material (steel) (not to scale)

→ For materials, whose yield stress is not well defined, it is defined as the value of stress producing 0.2% permanent strain

Stress-strain curve for ductile materials

Examples of ductile materials → steel, cast iron, aluminum

- **Strain hardening (Inelastic)**

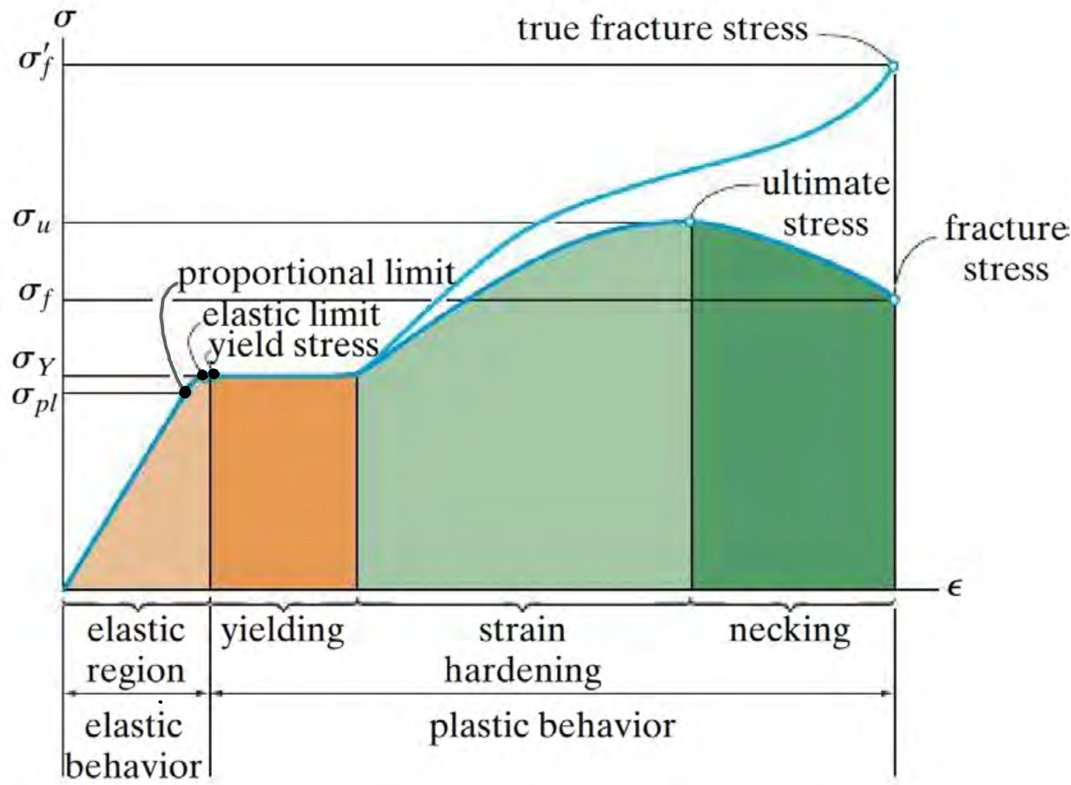
  When yielding has ended, an increase in load can be supported by the material resulting in a curve that rises continuously but becomes flatter until it reaches a maximum stress called ultimate stress

- **Necking (Inelastic)**

  Upto the ultimate stress, as the specimen elongates, its C/s area decreases uniformly

  But, just after the ultimate stress, the C/s area will begin to decrease in localized region

  As a result, a "neck" tends to form as the specimen elongates further and the break at fracture stress



Conventional and true stress-strain diagrams for ductile material (steel) (not to scale)

Necking

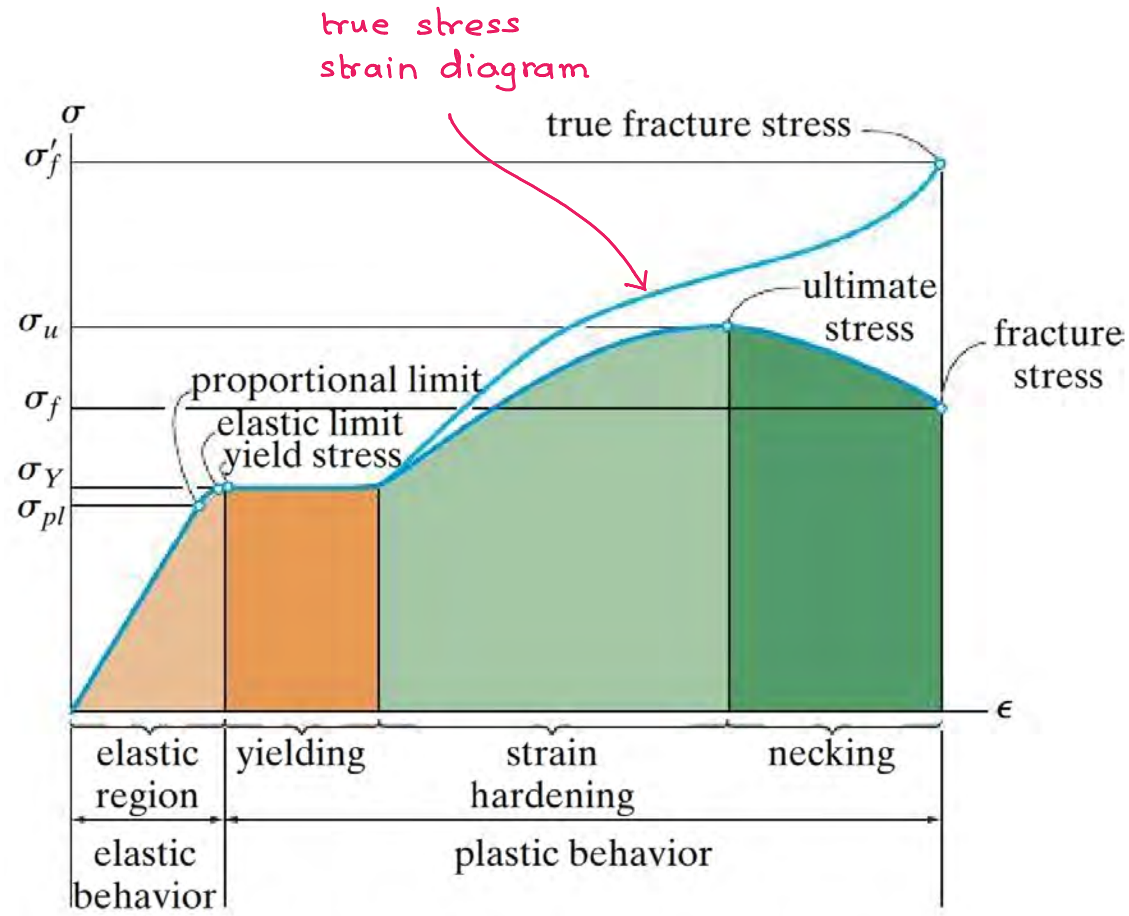
Failure of a ductile material

True stress-strain diagram

Often in experimental testing, one uses the original c/s area and specimen length to calculate the engineering stress and strain

Instead of that, one can also use the actual c/s area and specimen length at every instant of loading

The values of stress and strain found from these measurements are called true stress and true strain, and a plot of their values is called true stress-strain diagram

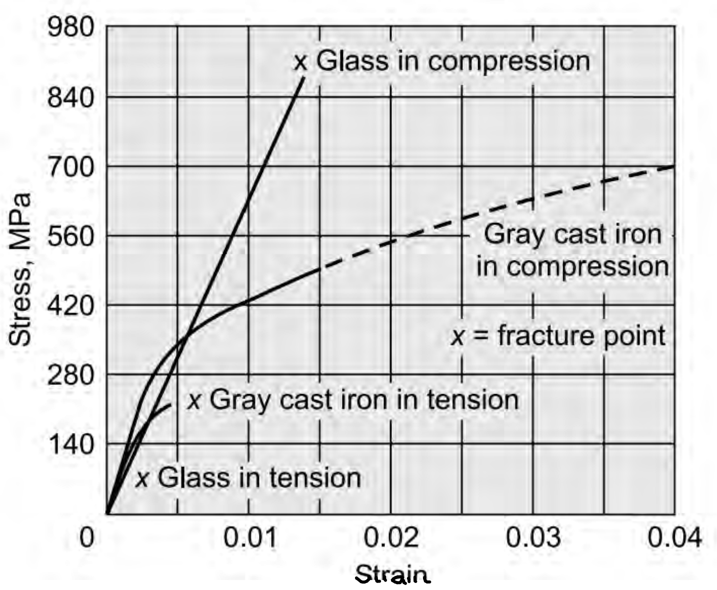


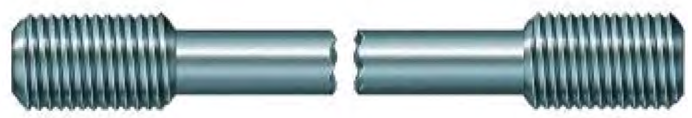
Conventional and true stress-strain diagrams for ductile material (steel) (not to scale)

Stress-strain curve for brittle materials

Brittle materials exhibit little or no yielding before failure

Example: Gray cast iron, concrete (in tension), glass





Tension failure of
a brittle material

- In general, it can be said that most materials exhibit both ductile and brittle behavior

- At low temperatures, materials become harder and more brittle

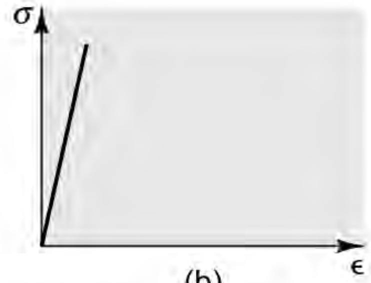
Idealizations of stress-strain curves

- In any problem of deformable bodies, you need to know the stress-strain relation
- (Stress-strain relation + equilibrium equations + strain-displacement) must be satisfied at every point in a deformable body in equilibrium
- Different materials often have quite dissimilar stress-strain relations
  - → No simple mathematical equation can fit the entire stress-strain curve of any material
  - → However, we want our mathematical analysis to be as simple as possible
  - → So, we idealize the stress-strain curves into forms which can be described by simple equations
  - → The kind of idealizations we make/choose will depend upon the magnitude of the strains that may arise in the problem

Some idealized stress-strain curves

- Rigid material – no strain regardless of stress
  - useful in studying gross motions and forces on machine parts

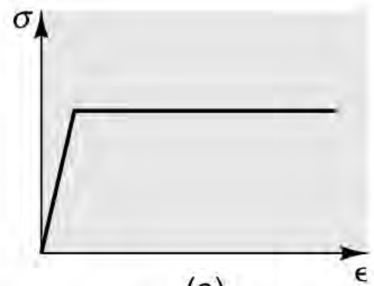
(a)
Rigid material



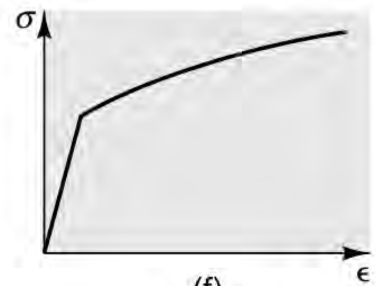
(b)
Linearly *elastic* material

- Linear elastic – strain is proportional to stress
  - useful when designing for small deformations

- Elastic-perfectly-plastic – both elastic and plastic strains are present, with negligible strain hardening
  - useful for designing bodies with large deformations



(e)
Elastic-perfectly plastic material
(non-strain-hardening)



(f)
Elastic-plastic material
(strain-hardening)

- Elasto-plastic – both elastic and plastic strains are present, with strain-hardening

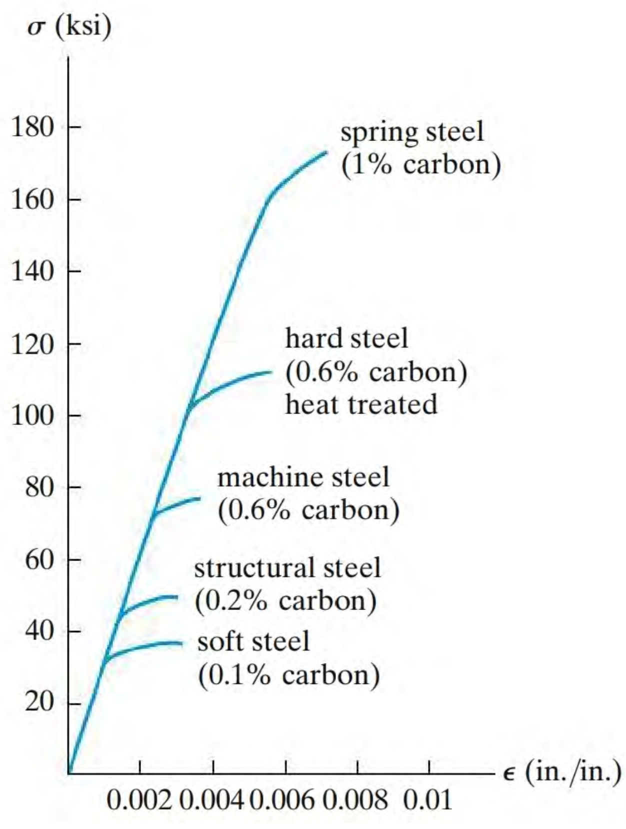
Linear Elastic Stress-Strain Relations

- From uniaxial tensile tests, it is found that most engineering materials exhibit a linear relationship between stress and strain within the elastic region

- We shall restrict ourselves to materials that are linear elastic in this course

- We need to relate all six stress components to all six strain components linearly

$$\underbrace{\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{bmatrix}}_{\text{Voigt notation}} = \underbrace{\begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1113} & C_{1112} \\ C_{2211} & C_{2222} & C_{2233} & C_{2223} & C_{2213} & C_{2212} \\ C_{3311} & C_{3322} & C_{3333} & C_{3323} & C_{3313} & C_{3312} \\ C_{2311} & C_{2322} & C_{2333} & C_{2323} & C_{2313} & C_{2312} \\ C_{1311} & C_{1322} & C_{1333} & C_{1323} & C_{1313} & C_{1312} \\ C_{1211} & C_{1222} & C_{1233} & C_{1223} & C_{1213} & C_{1212} \end{bmatrix}}_{\text{36 unknown elastic constants}} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \epsilon_{23} \\ \epsilon_{13} \\ \epsilon_{12} \end{bmatrix}$$



Linear Elastic Stress-Strain Relations

- From internal stored energy function, it can be proved that the elastic constants C_{ijkl} has major symmetry

$$\boxed{C_{ijkl} = C_{klij}}$$

- From : 36 constants $\xrightarrow[\text{symmetry}]{\text{Major}}$ 21 constants

$$\underbrace{\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{bmatrix}}_{\text{Voigt notation}} = \underbrace{\begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix}}_{\text{21 unknown elastic constants}} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \epsilon_{23} \\ \epsilon_{13} \\ \epsilon_{12} \end{bmatrix}$$

- These linear elastic materials with 21 elastic constants are called ANISOTROPIC materials