Local average rotation tensor

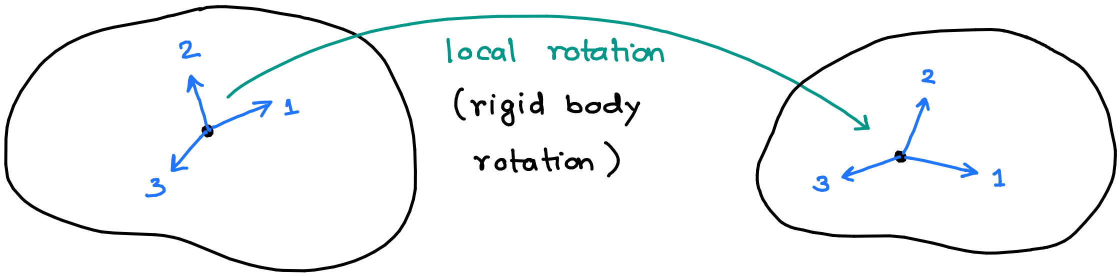
Note that the displacement gradient $\nabla \underline{u}$ can be written as:

$$\nabla \underline{u} = \frac{1}{2}\left(\nabla \underline{u} + \nabla \underline{u}^T\right) + \frac{1}{2}\left(\nabla \underline{u} - \nabla \underline{u}^T\right)$$

$$= \underline{\underline{\epsilon}} + \underline{\underline{W}}$$

small strain tensor (induces strain at a point)

local average rotation tensor (causes rigid-body rotation of line elements at a pt)



$$[\underline{\underline{W}}]_{\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}} = \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) \\ -\frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) & 0 & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}\right) \\ -\frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) & -\frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}\right) & 0 \end{bmatrix}$$

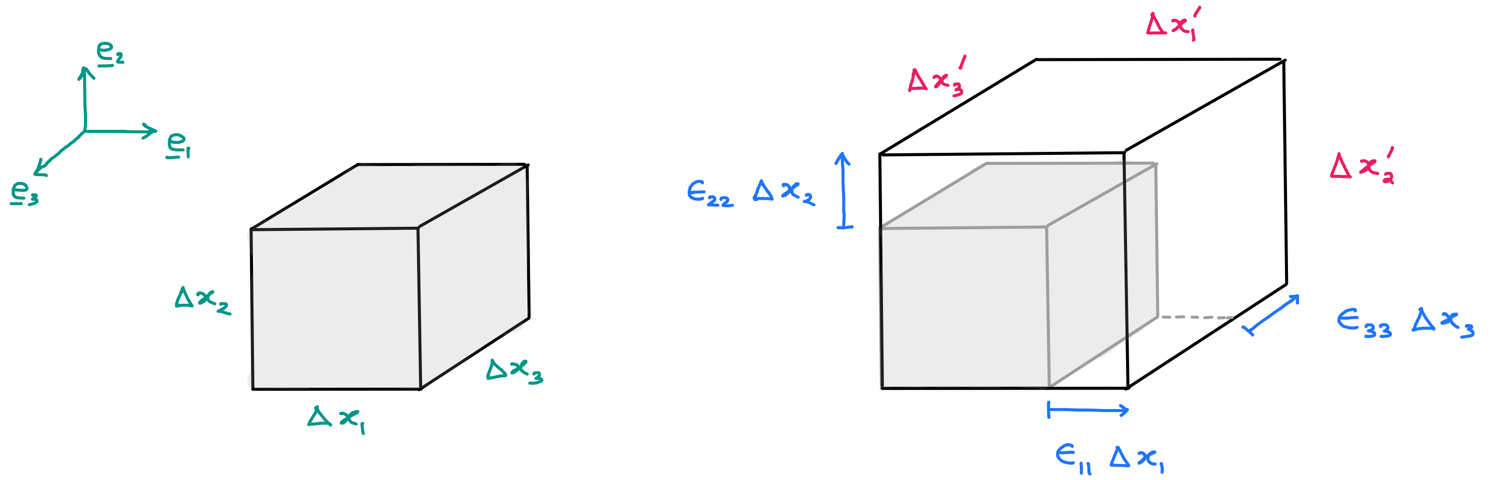
ANTI-SYMMETRIC MATRIX

The local average rotation tensor is responsible for rigid-body rotation of line elements. So if the displacement is such that the strain tensor $\underline{\underline{\epsilon}}$ is $\underline{\underline{0}}$ at a point, then there will be no strain of any kind (normal / shear strain) at that point. However, due to $\underline{\underline{W}}$, the line elements may undergo rigid rotation. $\underline{\underline{W}}$ can vary from point to point, meaning rigid body rotation will be different for different points in the body.

Local Volumetric Strain (or Dilatational Strain)

As a body deforms, the volume of every small region (called local volume element) of the body also changes. We can then define a quantity called volumetric strain because the change in volume per unit volume will be different for different parts in a body.

Normal strain leads to change in volume, so we will consider a small local volume element (in the form of a cuboid) at a point in the undeformed body.



$$\left.\begin{aligned} \Delta x_1' &= \Delta x_1 + \epsilon_{11}\,\Delta x_1 \\ \Delta x_2' &= \Delta x_2 + \epsilon_{22}\,\Delta x_2 \\ \Delta x_3' &= \Delta x_3 + \epsilon_{33}\,\Delta x_3 \end{aligned}\right\} \quad \text{for small strains}$$

Volume of original local volume element, $V = \Delta x_1\,\Delta x_2\,\Delta x_3$

Volume of the element after deformation, $v = \Delta x_1'\,\Delta x_2'\,\Delta x_3'$

$$\text{Volumetric strain} = \frac{v - V}{V} = \frac{(1+\epsilon_{11})(1+\epsilon_{22})(1+\epsilon_{33})\,V - V}{V}$$

$$\approx \epsilon_{11} + \epsilon_{22} + \epsilon_{33} \quad \text{(neglecting products of small strains)}$$

Strain-compatibility equations

We have seen that displacement of a point in a deformable body is a vector $\underline{u}$ which has three components.

$$[\underline{u}]_{\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

The deformation at a point is specified by six independent strain components

$$\epsilon_{11}, \ \epsilon_{22}, \ \epsilon_{33}, \ \epsilon_{12}, \ \epsilon_{13}, \ \epsilon_{23}$$

We have seen from strain-displacement relations that the six strains can be determined from three displacement functions, since they only involve differentiation of the displacement. Example:

$$\epsilon_{11} = \frac{\partial u_1}{\partial x_1}, \quad \epsilon_{22} = \frac{\partial u_2}{\partial x_2}, \quad \epsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$$

However, the reverse operation, i.e. determination of the three displacement functions u_1, u_2, u_3 from six strain components is not straightforward. Infact, we may not obtain a consistent displacement function by integrating any six arbitrary strain components

For example, think of integrating $\epsilon_{11}(x_1, x_2, x_3)$ w.r.t x_1 to obtain u_1

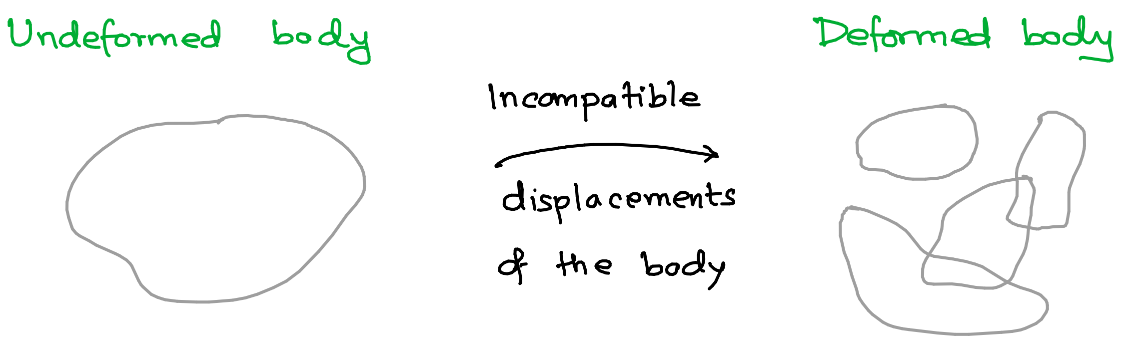
$$\Rightarrow u_1(x_1, x_2, x_3) = \int \epsilon_{11}(x_1, x_2, x_3)\, dx_1 + c_1(x_2, x_3)$$

Then integrating $\epsilon_{22}(x_1, x_2, x_3)$ w.r.t x_2 should give u_2

$$\Rightarrow u_2(x_1, x_2, x_3) = \int \epsilon_{22}(x_1, x_2, x_3)\, dx_2 + c_2(x_1, x_3)$$

The resulting functions u_1 and u_2 obtained from integration of ϵ_{11} and ϵ_{22} maynot satisfy the prescribed function for ϵ_{12}

Since 6 strain components are derived from 3 displacements, the six strain components can be arbitrary, else six arbitrary strain components could lead to "incompatible" displacement. Physically, it may happen that the displacement so obtained is such that it leads to two parts of a body overlapping with each other or getting separated



Thus, there has to be some constraint on the six strain functions which are collectively called STRAIN COMPATIBILITY conditions

Remember that a general symmetric matrix does not necessarily represent a strain matrix until it satisfies strain compatibility conditions.

There are SIX strain-compatibility equations (much like three stress-equilibrium relations), which can be divided into two sets:

$$\boxed{\text{Set } 1} \quad \left\{ \begin{aligned} \frac{\partial^2 \epsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \epsilon_{22}}{\partial x_1^2} &= 2\,\frac{\partial^2 \epsilon_{12}}{\partial x_1 \partial x_2} \\ \frac{\partial^2 \epsilon_{22}}{\partial x_3^2} + \frac{\partial^2 \epsilon_{33}}{\partial x_2^2} &= 2\,\frac{\partial^2 \epsilon_{23}}{\partial x_2 \partial x_3} \\ \frac{\partial^2 \epsilon_{11}}{\partial x_3^2} + \frac{\partial^2 \epsilon_{33}}{\partial x_1^2} &= 2\,\frac{\partial^2 \epsilon_{13}}{\partial x_1 \partial x_3} \end{aligned} \right.$$

We can verify the relations of set 1 by plugging in strain-disp relation.

$$\text{LHS} = \frac{\partial^2 \epsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \epsilon_{22}}{\partial x_1^2} = \frac{\partial^3 u_1}{\partial x_2^2 \partial x_1} + \frac{\partial^3 u_2}{\partial x_1^2 \partial x_2}$$

$$= \frac{\partial^2}{\partial x_1 \partial x_2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = 2\frac{\partial^2 \epsilon_{12}}{\partial x_1 \partial x_2} = \text{RHS}$$

Similarly, you can prove others

The second set of compatibility conditions is:

$$\frac{\partial}{\partial x_3}\left(\frac{\partial \epsilon_{23}}{\partial x_1} + \frac{\partial \epsilon_{31}}{\partial x_2} - \frac{\partial \epsilon_{12}}{\partial x_3}\right) = \frac{\partial^2 \epsilon_{33}}{\partial x_1 \partial x_2}$$

$$\frac{\partial}{\partial x_1}\left(\frac{\partial \epsilon_{12}}{\partial x_3} + \frac{\partial \epsilon_{13}}{\partial x_2} - \frac{\partial \epsilon_{23}}{\partial x_1}\right) = \frac{\partial^2 \epsilon_{11}}{\partial x_2 \partial x_3}$$

$$\frac{\partial}{\partial x_2}\left(\frac{\partial \epsilon_{12}}{\partial x_3} + \frac{\partial \epsilon_{23}}{\partial x_1} - \frac{\partial \epsilon_{13}}{\partial x_2}\right) = \frac{\partial^2 \epsilon_{33}}{\partial x_1 \partial x_2}$$

Special case for plane strain

For the case where strains are significant only on a plane, say $e_1 - e_2$ (or x-y) plane, we can neglect strains along the e_3 directions, i.e.

$$\left.\begin{aligned} \epsilon_{11} &= \epsilon_{11}(x_1, x_2) \\ \epsilon_{22} &= \epsilon_{22}(x_1, x_2) \\ \epsilon_{12} &= \epsilon_{12}(x_1, x_2) \end{aligned}\right\} \text{functions of } x_1 \text{ and } x_2 \text{ only}$$

$$\epsilon_{33} = \epsilon_{13} = \epsilon_{23} = 0$$

For this case, five compatibility conditions are automatically satisfied. Only $\frac{\partial^2 \epsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \epsilon_{22}}{\partial x_1^2} = 2\frac{\partial^2 \epsilon_{12}}{\partial x_1 \partial x_2}$ needs checking.