Stress Invariants

Recall the cubic equation we obtained for the stress eigenvalue problem

$$[\underline{\underline{\sigma}}][\underline{n}] = \lambda [\underline{n}]$$

$$\Rightarrow \det\left([\underline{\underline{\sigma}}] - \lambda [\underline{\underline{I}}] \right) = 0$$

$$\Rightarrow -\lambda^3 + I_1 \lambda^2 - I_2 \lambda + I_3 = 0$$

STRESS INVARIANTS

$$\begin{cases} I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33} \\[2ex] I_2 = \begin{vmatrix} \sigma_{11} & \tau_{12} \\ \tau_{12} & \sigma_{22} \end{vmatrix} + \begin{vmatrix} \sigma_{22} & \tau_{23} \\ \tau_{23} & \sigma_{33} \end{vmatrix} + \begin{vmatrix} \sigma_{11} & \tau_{13} \\ \tau_{13} & \sigma_{33} \end{vmatrix} \\[2ex] I_3 = \begin{vmatrix} \sigma_{11} & \tau_{12} & \tau_{13} \\ \tau_{12} & \sigma_{22} & \tau_{23} \\ \tau_{13} & \tau_{23} & \sigma_{33} \end{vmatrix} \end{cases}$$

$I_1 \rightarrow$ 1st stress invariant

$I_2 \rightarrow$ 2nd stress invariant

$I_3 \rightarrow$ 3rd stress invariant

These values donot change with the choice of coordinate system

In other words, if you choose a different coordinate system $\hat{\underline{e}}_1 - \hat{\underline{e}}_2 - \hat{\underline{e}}_3$, where the stress components are say $\hat{\sigma}_{11}, \hat{\sigma}_{22}, \hat{\sigma}_{33}, \hat{\tau}_{12}, \hat{\tau}_{13}, \hat{\tau}_{23}$, still the invariants will remain the same.

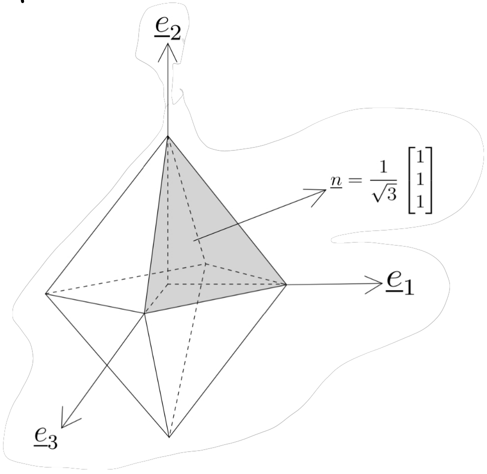
e.g. $\quad \hat{I}_1 = \hat{\sigma}_{11} + \hat{\sigma}_{22} + \hat{\sigma}_{33} = I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33}$

Using principal stresses

$$I_1 = \lambda_1 + \lambda_2 + \lambda_3$$
$$I_2 = \lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_1 \lambda_3$$
$$I_3 = \lambda_1 \lambda_2 \lambda_3$$

Octahedral Stresses

First we need to know what an octahedral plane is : they are faces of an octahedron having 8 faces whose normals have the form



$$[\underline{n}] = \begin{bmatrix} \pm 1/\sqrt{3} \\ \pm 1/\sqrt{3} \\ \pm 1/\sqrt{3} \end{bmatrix}$$

in the coordinate system of principal directions i.e. they are equally inclined from all principal directions.

The normal and shear components of traction on the octahedral faces are called octahedral stress components. The normal component of traction on octahedral planes will be

$$\sigma_{oct} = ([\underline{\sigma}][\underline{n}]) \cdot [\underline{n}]$$

$$= \left(\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} \pm 1/\sqrt{3} \\ \pm 1/\sqrt{3} \\ \pm 1/\sqrt{3} \end{bmatrix} \right) \cdot \begin{bmatrix} \pm 1/\sqrt{3} \\ \pm 1/\sqrt{3} \\ \pm 1/\sqrt{3} \end{bmatrix}$$

$$= \frac{\lambda_1 + \lambda_2 + \lambda_3}{3} = \frac{I_1}{3} \leftarrow \text{invariant}$$

For stress matrix expressed in a general coordinate system,

$$[\underline{\sigma}] = \begin{bmatrix} \sigma_{11} & \tau_{12} & \tau_{13} \\ & \sigma_{22} & \tau_{23} \\ \text{sym} & & \sigma_{33} \end{bmatrix}, \qquad \sigma_{oct} = \frac{I_1}{3} = \frac{\sigma_{11} + \sigma_{22} + \sigma_{33}}{3}$$

We can also find the shear stress component on octahedral planes

$$\tau_{oct}^2 = \| \underline{\underline{\sigma}}\, \underline{n} \|^2 - \sigma_{oct}^2$$

$$= \left\| \begin{bmatrix} \pm \lambda_1/\sqrt{3} \\ \pm \lambda_2/\sqrt{3} \\ \pm \lambda_3/\sqrt{3} \end{bmatrix} \right\|^2 - \left(\frac{\lambda_1 + \lambda_2 + \lambda_3}{3} \right)^2$$

$$= \frac{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}{3} - \frac{(\lambda_1 + \lambda_2 + \lambda_3)^2}{9}$$

$$= \frac{2(\lambda_1 + \lambda_2 + \lambda_3)^2 - 6(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_4)}{9}$$

$$= \frac{2 I_1^2 - 6 I_2}{9}$$

$$\Rightarrow \tau_{oct} = \frac{\sqrt{2}}{3} \sqrt{I_1^2 - 3 I_2} \quad \Bigg\}$$

turns out invariant as well since I_1^2 and I_2 are invariants

Key points : a) Octahedral planes are defined w.r.t principal directions and not with any arbitrary frame of reference i.e.

$$[\underline{n}]_{(n_1, n_2, n_3)} = \begin{bmatrix} \pm 1/\sqrt{3} \\ \pm 1/\sqrt{3} \\ \pm 1/\sqrt{3} \end{bmatrix}$$

b) We have the same σ_{oct} and τ_{oct} on all 8 faces of the octahedron

c) If the first invariant $I_1 = 0$, then normal stresses on the octahedral planes are zero and only shear acts

State of pure shear

The state of stress at a point can be characterized by 6 independent stress components. The magnitudes of these components depend upon the choice of the coordinate sys.

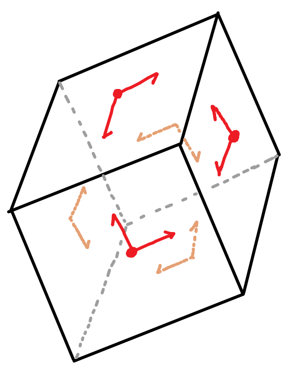
If, for at least one particular choice of coordinate system, we find that $\sigma_{11} = \sigma_{22} = \sigma_{33} = 0$, then a state of pure shear is said to exist at that point. In that particular coordinate system, the stress matrix will be

$$[\underline{\underline{\sigma}}] = \begin{bmatrix} 0 & \tau_{12} & \tau_{13} \\ \tau_{12} & 0 & \tau_{23} \\ \tau_{13} & \tau_{23} & 0 \end{bmatrix}$$

For this coordinate system, $I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33} = 0$.

But since I_1 is an invariant and does not depend upon the choice of coordinate system, $I_1 = 0$ for any choice of coordinate system. Hence, the condition needed for state of pure shear to exist at a point is to have $I_1 = 0$

State of pure shear in some coordinate system →



Hydrostatic and Deviatoric (pure shear) parts of stress tensor

An arbitrary state of stress can be decomposed into a hydrostatic state of stress and a $\underbrace{\text{state of pure shear}}_{\text{deviatoric}}$.

$$\underline{\underline{\sigma}} = \underbrace{\frac{1}{3} \overbrace{I_1(\underline{\underline{\sigma}})}^{\text{scalar value}} \underline{\underline{I}}}_{\underline{\underline{\sigma}}_{hyd}} + \underbrace{\left(\underline{\underline{\sigma}} - \frac{1}{3} I_1(\underline{\underline{\sigma}})\right)}_{\underline{\underline{\sigma}}_{dev}}$$

The hydrostatic stress tensor $\frac{1}{3} I_1(\underline{\underline{\sigma}})\, \underline{\underline{I}}$ is proportional to the identity tensor $\underline{\underline{I}}$. The reason it is called hydrostatic is because it has same normal stress acting in all three faces, just like pressure acting at a point in a static fluid.

$p = I_1/3$

$$\begin{bmatrix} \sigma_{11} & \tau_{12} & \tau_{13} \\ \tau_{12} & \sigma_{22} & \tau_{23} \\ \tau_{13} & \tau_{23} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix} + \begin{bmatrix} \sigma_{11}-p & \tau_{12} & \tau_{13} \\ \tau_{12} & \sigma_{22}-p & \tau_{23} \\ \tau_{13} & \tau_{23} & \sigma_{33}-p \end{bmatrix}$$

The deviatoric part has first stress invariant as zero

$$\begin{aligned} I_1(\underline{\underline{\sigma}}_{dev}) &= (\sigma_{11}-p) + (\sigma_{22}-p) + (\sigma_{33}-p) \\ &= \sigma_{11} + \sigma_{22} + \sigma_{33} - 3p \\ &= \sigma_{11} + \sigma_{22} + \sigma_{33} - 3\left(\frac{\sigma_{11}+\sigma_{22}+\sigma_{33}}{3}\right) = 0 \end{aligned}$$

The hydrostatic stress tries to change the size of cuboid without distorting the shape while the deviatoric stress tries to distort the shape of cuboid without changing the size/volume of cuboid