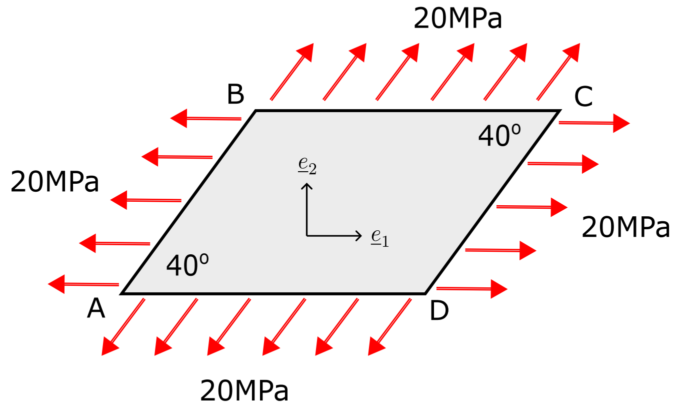
Minor

Total marks: 30 **Total time**: 2 hours

Answers without proper reasoning/motivation will be given no partial marks.

1. [**5 marks**] An angular plate in a state of plane stress is subjected to uniform tensile pressure of 20 MPa on its sides as shown in the figure below.

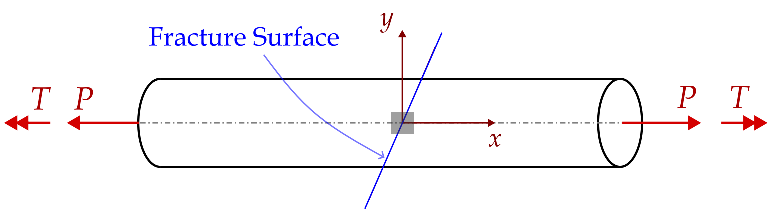


If the state of stress is uniform (i.e. it does not vary from point to point) throughout the plate, determine

(a) the traction vectors (expressed in $e_1 - e_2$ coordinate system) on planes AB and BC,

(b) the stress components σ_{11}, σ_{22}, and τ_{12} at any point in the plate.

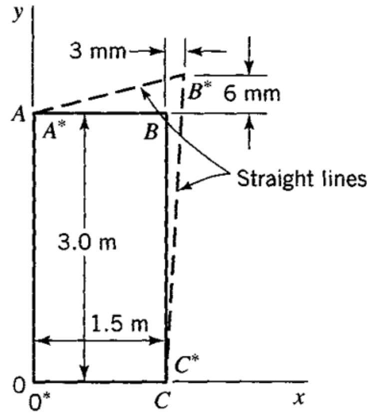
2. [**8 marks**] A piece of chalk is subjected to combined loading consisting of a tensile load P and a torque T. The chalk has an ultimate threshold strength σ_u, as determined in a simple tensile test. The load P remains constant at such a value that it produces a tensile stress $0.5\sigma_u$, on any cross-section (perpendicular to x-axis). The torque T is increased gradually until the chalk fractures off along some inclined plane originating at the outer surface of the chalk.



Assuming that fracture takes place when the maximum principal stress reaches the ultimate threshold strength, determine the magnitude of the torsional shear stress produced by torque T at fracture and determine the orientation of the fracture plane at the surface of the chalk **with respect to the x-axis**.

3. [**2 marks**] Show that the principal directions of the deviatoric part of stress tensor $\underline{\underline{\sigma}}_{\text{dev}}$ is the same as those of the stress tensor $\underline{\underline{\sigma}}$ itself.

4. **[9 marks]** A rectangular panel (as shown below) on the body of a space shuttle is loaded in such a fashion that it can be assumed a state of plane strain exists ($\epsilon_{zz} = \epsilon_{zx} = \epsilon_{zy} = 0$).



The deformed configuration of the rectangular panel is shown in dotted lines.

(a) Two sides of the rectangular panel undergo no displacements since they are part of boundary conditions. Identify these two sides and the type of boundary conditions imposed (Neumann or Dirichlet).

(b) Determine the expressions of displacement components throughout the panel.

(c) Determine the strain components at point B.

(d) Which point in the panel will have the maximum normal strain and what will be its orientation?

5. **[6 marks]** Consider the strain components in the plate ABCD, in terms of the coordinate system (x, y)

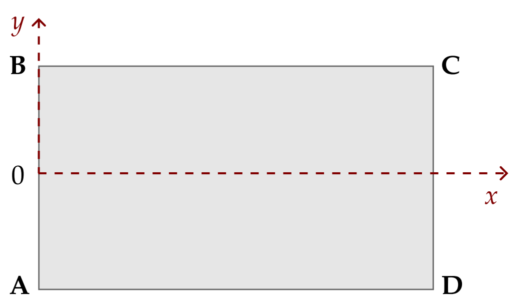
$$\epsilon_{xx} = Cy(L - x), \quad \epsilon_{yy} = Dy(L - x), \quad \gamma_{xy} = -(C + D)(A^2 - y^2)$$

where A, C, and D are known constants. CHECK if the strain field is compatible or not. The displacement components (u_x, u_y) at $x = y = 0$ are

$$u_x(0, 0) = 0, \quad u_y(0, 0) = 0$$

and slope $\frac{\partial u_x}{\partial y}$ at $x = y = 0$ is

$$\left.\frac{\partial u_x}{\partial y}\right|_{x=y=0} = 0$$



Determine u_x and u_y as functions of x and y.

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Determine u_x and u_y as functions of x and y.

For plane strain: $\dfrac{\partial^2 \epsilon_{xx}}{\partial y^2} + \dfrac{\partial^2 \epsilon_{yy}}{\partial x^2} = \dfrac{\partial^2 \gamma_{xy}}{\partial x \, \partial y}$ (Satisfied) 1

Soln:

0.5 $\quad \boxed{\dfrac{\partial u_x}{\partial x} = \epsilon_{xx}} = Cy(L-x) \quad$ — (a)

0.5 $\quad \boxed{\dfrac{\partial u_y}{\partial y} = \epsilon_{yy}} = Dy(L-x) \quad$ — (b)

1 $\quad \boxed{\dfrac{\partial u_x}{\partial y} + \dfrac{\partial u_y}{\partial x}} = \gamma_{xy} = -(C+D)(A^2 - y^2) \quad$ — (c)

Integration of (a) and (b)

1 $\quad u_x = Cy\left(Lx - \dfrac{x^2}{2}\right) + f(y) \quad$ — (d)

1 $\quad u_y = \dfrac{1}{2} D y^2 (L-x) + g(x) \quad$ — (e)

where $f(y)$ and $g(x)$ are functions of y and x respectively.

Substituting (d) and (e) in (c) :

$$\left[C\left(Lx-\frac{x^2}{2}\right)+f'(y)\right]+\left[-\frac{1}{2}Dy^2+g'(x)\right]$$

$$=-(C+D)(A^2-y^2)$$

$$\Rightarrow \quad C\left(Lx-\frac{x^2}{2}\right)+g'(x)=$$

$$\frac{1}{2}Dy^2-(C+D)(A^2-y^2)-f'(y)$$

(1) LHS is a function of x and RHS is a function of y, therefore both sides must be constant

$$C\left(Lx-\frac{x^2}{2}\right)+g'(x)=E_1 \text{ (constant)} \quad \text{— (f)}$$

$$\frac{1}{2}Dy^2-(C+D)(A^2-y^2)-f'(y)=E_1 \text{ (constant)} \quad \text{— (g)}$$

Integrating (f) and (g) :

$$g(x)=-C\left(\frac{1}{2}Lx^2-\frac{x^3}{6}\right)+E_1x+E_2$$

$$f(y)=\frac{1}{6}Dy^3-(C+D)\left(A^2y-\frac{y^3}{3}\right)-E_1y+E_3$$

Using these values, we can write the displacements as:

$$u_x = Cy\left(Lx - \frac{x^2}{2}\right) + \left(\frac{C}{3} + \frac{D}{2}\right) y^3 - \left(E_1 + (C+D)A^2\right) y + E_3$$

$$u_y = \frac{1}{2} D y^2 (L - x) - C\left(\frac{1}{2} L x^2 - \frac{x^3}{6}\right) + E_1 x + E_2$$

Now, use boundary conditions:

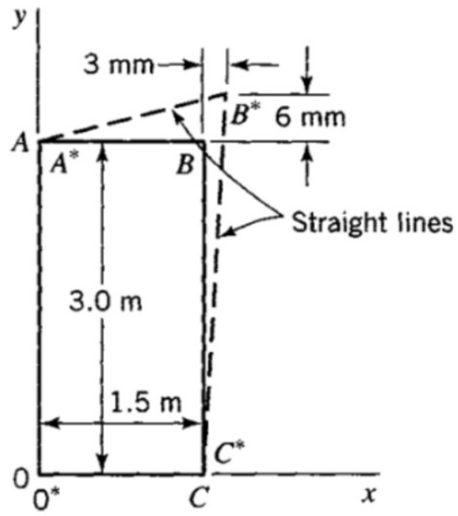
$$u_x(0,0) = 0 \Rightarrow E_3 = 0 \quad 0.5$$

$$u_y(0,0) = 0 \Rightarrow E_2 = 0 \quad 0.5$$

$$\left.\frac{\partial u_x}{\partial y}\right|_{\substack{x=0\\y=0}} = 0 \Rightarrow E_1 + (C+D)A^2 = 0$$

$$\Rightarrow E_1 = -(C+D)A^2 \quad 1$$

4. [**9 marks**] A rectangular panel (as shown below) on the body of a space shuttle is loaded in such a fashion that it can be assumed a state of plane strain exists ($\epsilon_{zz} = \epsilon_{zx} = \epsilon_{zy} = 0$).



The deformed configuration of the rectangular panel is shown in dotted lines.

(a) Two sides of the rectangular panel undergo no displacements since they are part of boundary conditions. Identify these two sides and the type of boundary conditions imposed (Neumann or Dirichlet).

(b) Determine the expressions of displacement components throughout the panel.

(c) Determine the strain components at point B.

(d) Which point in the panel will have the maximum normal strain and what will be its orientation?

Soln

(a) Sides OA and OC have Dirichlet BCs or displacement BCs 1

(b) We have to guess the displacement field!

$$u_x(x,y) = a_0 + a_1 x + a_2 y + a_3 xy + a_4 x^2 + a_5 y^2 + \cdots$$

$$u_y(x,y) = b_0 + b_1 x + b_2 y + b_3 xy + b_4 x^2 + b_5 y^2 + \cdots$$

- The edges remain straight before and after deformation

$\Rightarrow$ u_x and u_y will not have higher-order terms in x and y

$$\Rightarrow \quad u_x = a_0 + a_1 x + a_2 y + a_3 xy$$
$$u_y = b_0 + b_1 x + b_2 y + b_3 xy$$

1

- u_x and u_y along OC is zero

$$u_x\Big|_{\substack{x=x\\y=0}} = 0 \Rightarrow \begin{array}{l} a_0 = 0 \\ a_1 = 0 \end{array}$$

$$u_y\Big|_{\substack{x=x\\y=0}} = 0 \Rightarrow \begin{array}{l} b_0 = 0 \\ b_1 = 0 \end{array}$$

0.5

- u_x and u_y along OA is zero

$$u_x\Big|_{\substack{x=0\\y=y}} = 0 \Rightarrow a_2 = 0$$

$$u_y\Big|_{\substack{x=0\\y=y}} = 0 \Rightarrow b_2 = 0$$

0.5

- $u_x\,(x = 1.5,\ y = 3) = 0.003$ m

$$\Rightarrow \quad a_3 = \frac{0.003}{1.5 \times 3} = \frac{2}{3} \times 10^{-3}$$

0.5

$$u_y(x=1.5,\ y=3) = 0.006\ \text{m}$$

$$\Rightarrow \quad b_3 = \frac{0.006}{1.5 \times 3} = \frac{4}{3} \times 10^{-3} \qquad 0.5$$

So finally we get

$$u_x = 2/3 \times 10^{-3}\ xy$$

$$u_y = 4/3 \times 10^{-3}\ xy \qquad 1$$

(c) At point B, the strain components are:

$$\epsilon_{xx}\Big|_B = \frac{\partial u_x}{\partial x}\Big|_{\substack{x=1.5\\ y=3}} = 2/3 \times 10^{-3}\ y = 2/3 \times 10^{-3}(3)$$

$$= 2 \times 10^{-3}$$

1

$$\epsilon_{yy}\Big|_B = \frac{\partial u_y}{\partial y}\Big|_{\substack{x=1.5\\ y=3}} = 4/3 \times 10^{-3}\ x = 4/3 \times 10^{-3}(1.5)$$

$$= 2 \times 10^{-3}$$

$$\epsilon_{xy}\Big|_B = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)\Big|_{\substack{x=1.5\\ y=3}}$$

$$= \frac{1}{2}\left(\frac{2}{3}x + \frac{4}{3}y\right) \times 10^{-3}$$

1

$$= \left[\frac{1}{3}(1.5) + \frac{2}{3}(3)\right] \times 10^{-3} = 2.5 \times 10^{-3}$$

Equivalently, one could also report γ_{xy}

$$\gamma_{xy} = 2\,\epsilon_{xy} = 5 \times 10^{-3}$$

(d) The maximum normal strain can be found either by using different methods:

(i) Formula based or (ii) Mohr's circle-based

easier way

$$\text{Center} = \left(\frac{\epsilon_{xx} + \epsilon_{yy}}{2}, 0 \right)$$

$$= \left(\frac{\frac{2}{3}y + \frac{4}{3}x}{2}, 0 \right)$$

Strain components on x-plane at any given pt. (x, y)

$$\epsilon_{xx} = \frac{2}{3}y \times 10^{-3}$$

$$\epsilon_{xy} = \frac{1}{2}\left(\frac{2}{3}x + \frac{4}{3}y\right) \times 10^{-3}$$

$$= \left(\frac{x}{3} + \frac{2y}{3}\right) \times 10^{-3}$$

$$\epsilon_{yy} = \frac{4}{3}x \times 10^{-3}$$

One can check the state of strain at points A, C, and B. Out of these three points, the maximum values of strain components occur at point B.

1

The maximum normal strain is obtained at pt B

To find the orientation of axis of maximum normal strain, we can use the Mohr's circle:

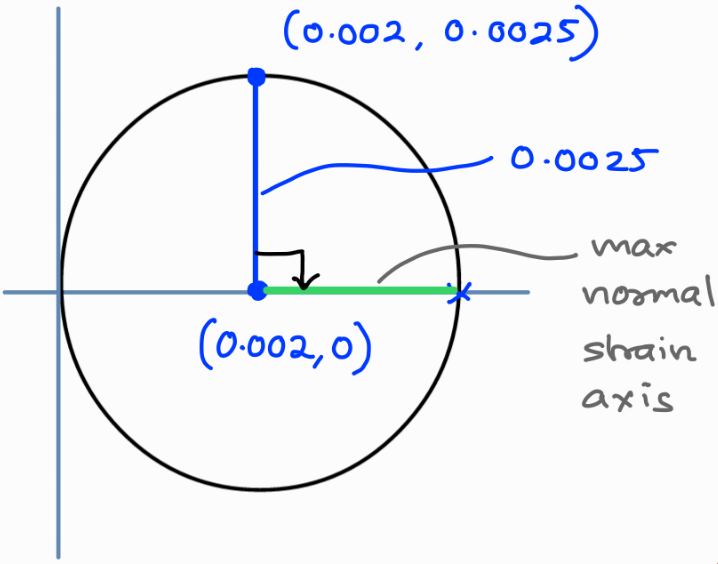
$$\epsilon_{xx} = 2 \times 10^{-3}$$

$$\epsilon_{yy} = 2 \times 10^{-3}$$

$$\epsilon_{xy} = 2.5 \times 10^{-3}$$

Center of Mohr's circle $= \left(\frac{\epsilon_{xx} + \epsilon_{yy}}{2}, 0 \right)$

$= 2 \times 10^{-3}$

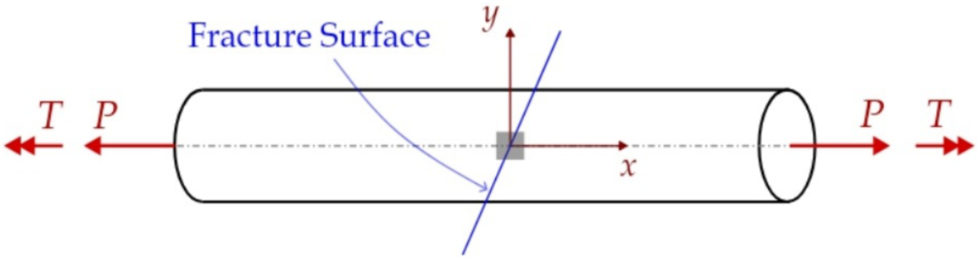


Radius $= 2.5 \times 10^{-3}$

Physically, the orientation is at 45° anticlockwise from x-axis at pt B

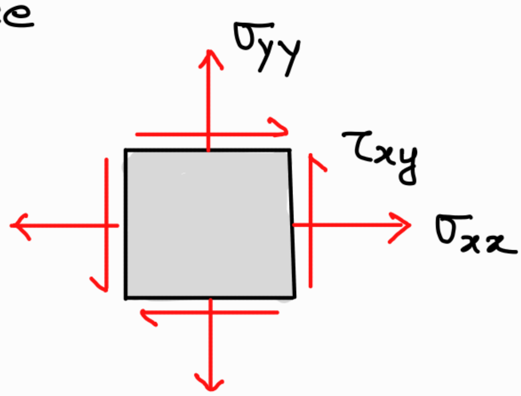
1

2. **[8 marks]** A piece of chalk is subjected to combined loading consisting of a tensile load P and a torque T. The chalk has an ultimate threshold strength σ_u, as determined in a simple tensile test. The load P remains constant at such a value that it produces a tensile stress $0.5\sigma_u$, on any cross-section (perpendicular to x-axis). The torque T is increased gradually until the chalk fractures off along some inclined plane originating at the outer surface of the chalk.



Assuming that fracture takes place when the maximum principal stress reaches the ultimate threshold strength, determine the magnitude of the torsional shear stress produced by torque T at fracture and determine the orientation of the fracture plane at the surface of the chalk **with respect to the x-axis.**

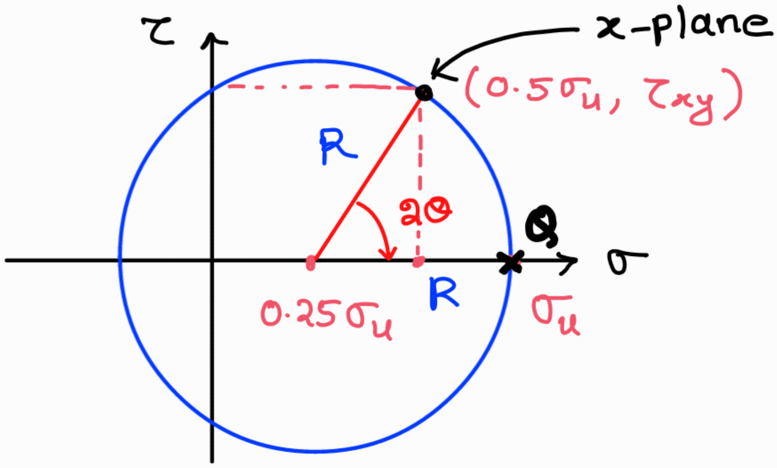
Soln Consider the plane-stress shaded element at the surface



- $\sigma_{xx} = 0.5\,\sigma_u$ — 1
- $\sigma_{yy} = 0$ — 1
- $\sigma_{max} = \sigma_u$

- $\tau_{xy} = ?$ (to be determined)
- Fracture plane orientation to be determined

Using Mohr's circle (or using stress-transformation formula), we can find the value of τ_{xy} as well as the direction of max principal stress.



$$\text{Center} \equiv \left(\frac{\sigma_{xx} + \sigma_{yy}}{2}, 0 \right)$$

$$\text{Radius} \equiv \sigma_u - 0.25\sigma_u$$
$$= 0.75\,\sigma_u$$

$$(0.5\,\sigma_u - 0.25\,\sigma_u)^2 + \tau_{xy}^2 = (0.75\,\sigma_u)^2$$

$$\Rightarrow \quad 0.25^2\,\sigma_u^2 + \tau_{xy}^2 = 0.75^2\,\sigma_u^2$$

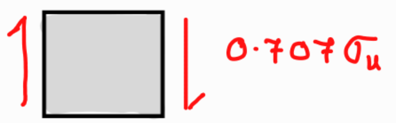
$$\Rightarrow \quad \tau_{xy}^2 = (0.75^2 - 0.25^2)\,\sigma_u^2$$

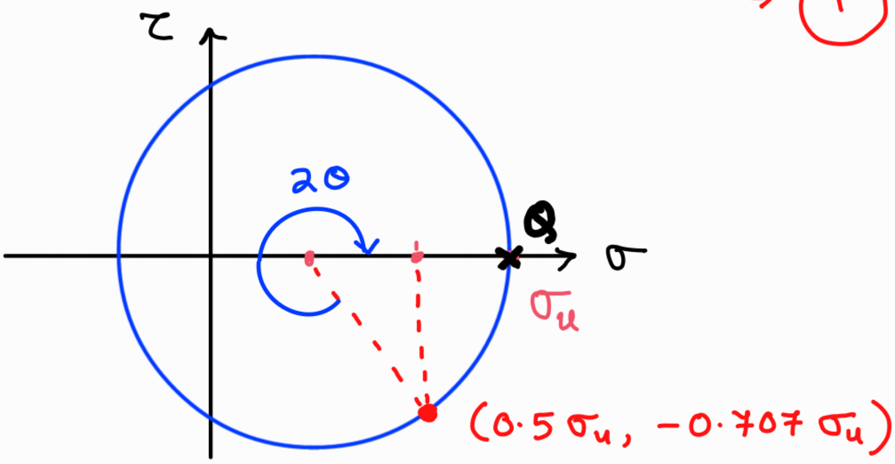
$$\Rightarrow \quad \tau_{xy} = \pm\sqrt{0.5}\,\sigma_u \quad — \ 2$$

Since the torque acting on the right end of the piece of chalk is counterclockwise, the shear stress τ_{xy} acts down on the +ve face of plane element and is therefore negative

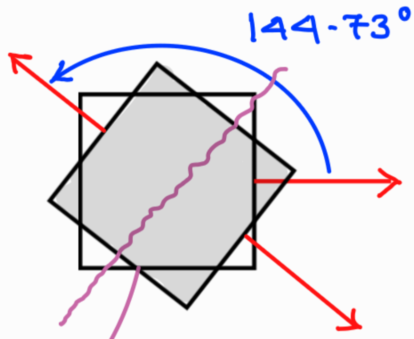
$$\tau_{xy} = -\,0.707\,\sigma_u$$

1





The point Q representing the max normal stress is located by rotating clockwise through 2θ from x-plane. Therefore, the plane of max normal stress is physically oriented at an angle of θ counterclockwise from x-plane normal.

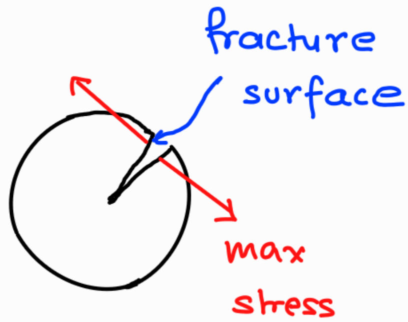


2

$$\tan 2\theta = \frac{(-0.707\,\sigma_u)}{0.25\,\sigma_u} = -2.828$$

$$\Rightarrow \theta = \begin{cases} -0.61546 \text{ rad} & (-35.26^\circ) \\ 2.52613 \text{ rad} & (144.73^\circ) \end{cases}$$

The orientation of fracture plane is going to be perpendicular to the



Thus, the fracture plane will be oriented at an angle of

$$\phi = \begin{cases} \theta + \frac{\pi}{2} \\ \theta - \pi/2 \end{cases}$$

1

3. **[2 marks]** Show that the principal directions of the deviatoric part of stress tensor $\underline{\underline{\sigma}}_{\text{dev}}$ is the same as those of the stress tensor $\underline{\underline{\sigma}}$ itself.

Solu:

$$\underline{\underline{\sigma}} = \frac{1}{3} I_1 \underline{\underline{I}} + \underbrace{\left(\underline{\underline{\sigma}} - \frac{1}{3} I_1 \underline{\underline{I}}\right)}_{\underline{\underline{\sigma}}_{dev}}$$

0.5

$$I_1 = \text{trace}(\underline{\underline{\sigma}})$$

$$\underline{\underline{\sigma}}\, \underline{n} = \underline{\lambda}\, \underline{n}$$

$$\Rightarrow \left(\frac{1}{3} I_1 \underline{\underline{I}} + \underline{\underline{\sigma}}_{dev}\right) \underline{n} = \lambda\, \underline{n}$$

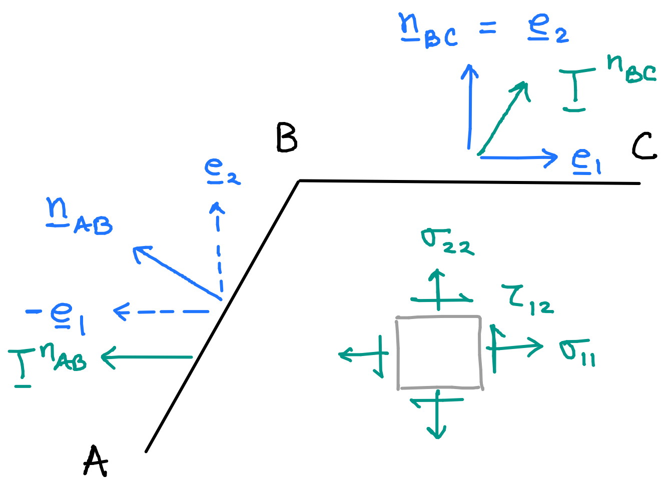
$$\Rightarrow \boxed{\underline{\underline{\sigma}}_{dev}\, \underline{n} = \underbrace{\left(\lambda - \frac{1}{3} I_1\right)}_{\text{scalar}} \underline{n}}$$

1.5

The above equation is the eigenvalue problem of the deviatoric part of stress tensor

The principal directions $\underline{n}$ remain the same for both $\underline{\underline{\sigma}}$ and $\underline{\underline{\sigma}}_{dev}$

1)



On plane BC

$$[\underline{T}^{n_{BC}}]_{(e_1, e_2)} = \begin{bmatrix} T_1^{n_{BC}} \\ T_2^{n_{BC}} \end{bmatrix}$$

$$T_1^{n_{BC}} = \underline{T}^{n_{BC}} \cdot \underline{e}_1 = 20 \cos 40^\circ = 15.32 \text{ MPa}$$

0.75

$$T_2^{n_{BC}} = \underline{T}^{n_{BC}} \cdot \underline{e}_2 = 20 \sin 40^\circ = 12.86 \text{ MPa}$$

0.75

On plane AB

$$[\underline{T}^{n_{AB}}]_{(e_1, e_2)} = \begin{bmatrix} T_1^{n_{AB}} \\ T_2^{n_{AB}} \end{bmatrix}$$

$$T_1^{n_{AB}} = \underline{T}^{n_{AB}} \cdot \underline{e}_1 = 20 \cos 180^\circ = -20 \text{ MPa}$$

0.75

$$T_2^{n_{AB}} = \underline{T}^{n_{AB}} \cdot \underline{e}_2 = 20 \cos 90^\circ = 0$$

0.75

For a uniform state of stress $[\underline{\sigma}]_{(e_1, e_2)} = \begin{bmatrix} \sigma_{11} & \tau_{12} \\ \tau_{12} & \sigma_{22} \end{bmatrix}$

$$[\underline{T}^{n_{AB}}] = [\underline{\sigma}][\underline{n}_{AB}] \quad , \quad [\underline{T}^{n_{BC}}] = [\underline{\sigma}][\underline{n}_{BC}]$$

where all vectors and matrices are expressed in $(e_1 - e_2)$ coor. sys

$$[\underline{n}_{AB}]_{\binom{e_1}{e_2}} = \begin{bmatrix} -\sin 40^\circ \\ \cos 40^\circ \end{bmatrix}, \qquad [\underline{n}_{BC}]_{\binom{e_1}{e_2}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

For $\underline{T}^{n_{BC}}$,

$$\begin{bmatrix} 15.32 \\ 12.86 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \tau_{12} \\ \tau_{12} & \sigma_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad \begin{aligned} \tau_{12} &= 15.32 \text{ MPa} \\ \sigma_{22} &= 12.86 \text{ MPa} \end{aligned} \quad \Bigg| \; 1$$

For $\underline{T}^{n_{AB}}$,

$$\begin{bmatrix} -20 \\ 0 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \tau_{12} \\ \tau_{12} & \sigma_{22} \end{bmatrix} \begin{bmatrix} -\sin 40^\circ \\ \cos 40^\circ \end{bmatrix}$$

$$\Rightarrow \quad \begin{aligned} \sigma_{11}(-\sin 40^\circ) + \tau_{12}\cos 40^\circ &= -20 \\ \tau_{12}(-\sin 40^\circ) + \sigma_{22}\cos 40^\circ &= 0 \end{aligned}$$

$$\sigma_{11} = \frac{-20 - \tau_{12}\cos 40^\circ}{-\sin 40^\circ} = 49.37 \text{ MPa} \quad \Big| \; 1$$