Q7) Find a relation between σ and ϵ using E and ν

since the result to be derived is expressed in terms of ν.

$$\epsilon_{rr} = \frac{1}{E}\left[\sigma_{rr} - \nu\sigma_{\theta\theta}\right]$$

$$\epsilon_{\theta\theta} = \frac{1}{E}\left[\sigma_{\theta\theta} - \nu\sigma_{rr}\right]$$

Because the temperature field is symmetrical w.r.t. the axis of the cylinder, it is valid to assume that the stresses and displacements are distributed as already provided in question, except that the strains ϵ_{rr} and $\epsilon_{\theta\theta}$ will now have temperature-induced strain as well

0.5

Due to a radial temperature increase $\Delta T(r)$, the total strains

$$\epsilon^{t}_{rr} = \epsilon_{rr} + \alpha\Delta T = \frac{1}{E}\left[\sigma_{rr} - \nu\sigma_{\theta\theta}\right] + \alpha\Delta T$$

$$\epsilon^{t}_{\theta\theta} = \epsilon_{\theta\theta} + \alpha\Delta T = \frac{1}{E}\left[\sigma_{\theta\theta} - \nu\sigma_{rr}\right] + \alpha\Delta T$$

1

Express the σ_{rr} and $\sigma_{\theta\theta}$ in terms of $\epsilon_{\theta\theta}$ and ϵ_{rr}

$$\sigma_{rr} = \frac{E}{1-\nu^2}\left[\epsilon_{rr} + \nu\epsilon_{\theta\theta} - (1+\nu)\alpha\Delta T\right]$$

$$\sigma_{\theta\theta} = \frac{E}{1-\nu^2}\left[\epsilon_{\theta\theta} + \nu\epsilon_{rr} - (1+\nu)\alpha\Delta T\right]$$

2.5

Next substitute σ_{rr} and $\sigma_{\theta\theta}$ in the radial equilibrium eqn:

$$\frac{d\sigma_{rr}}{dr} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0$$

$$\Rightarrow \frac{d}{dr}\left[\epsilon_{rr} + \nu\epsilon_{\theta\theta} - (1+\nu)\alpha\Delta T\right]$$

$$+\frac{1}{r}\left[\epsilon_{rr} + \nu\epsilon_{\theta\theta} - (1+\nu)\alpha\Delta T - \epsilon_{\theta\theta} - \nu\epsilon_{rr} + (1+\nu)\alpha\Delta T\right] = 0$$

$$\Rightarrow \frac{d\epsilon_{rr}}{dr} + \nu\frac{d\epsilon_{\theta\theta}}{dr} - (1+\nu)\alpha\frac{d\Delta T}{dr}$$

$$+\frac{1}{r}\left[(1-\nu)(\epsilon_{rr} - \epsilon_{\theta\theta})\right] = 0 \qquad (1)$$

Now use: $\epsilon_{rr} = \frac{du_r}{dr}$, $\epsilon_{\theta\theta} = \frac{u_r}{r}$

$$\Rightarrow \frac{d^2u_r}{dr^2} + \cancel{\frac{\nu}{r}\frac{du_r}{dr}} - \cancel{\nu\frac{u_r}{r^2}} + \frac{1}{r}\frac{du_r}{dr} - \cancel{\frac{\nu}{r}\frac{du_r}{dr}}$$

$$- \frac{u_r}{r^2} + \cancel{\nu\frac{u_r}{r^2}} = (1+\nu)\alpha\frac{d\Delta T}{dr}$$

$$\Rightarrow \frac{d^2u_r}{dr^2} + \frac{1}{r}\frac{du_r}{dr} - \frac{u_r}{r^2} = (1+\nu)\alpha\frac{d\Delta T}{dr}$$

$$\Rightarrow \frac{d}{dr}\left[\frac{1}{r}\frac{d(ru_r)}{dr}\right] = (1+\nu)\alpha\frac{d\Delta T}{dr} \qquad (1)$$

Q5) We write the strain energy of the system using

$$U_i = \frac{EI}{2} \int_0^L \left(\frac{d^2 v}{dx^2}\right)^2 dx$$

$$= \frac{EI}{2} \int_0^L \left[\sum_{n=1}^{\infty} a_n \left(\frac{n\pi}{L}\right)^2 \sin\left(\frac{n\pi x}{L}\right) \right]^2 dx$$

1

Expanding the terms in brackets;

1

$$\left[\sum_{n=1}^{\infty} a_n \left(\frac{n\pi}{L}\right)^2 \sin\left(\frac{n\pi x}{L}\right) \right]^2 = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_m a_n \left(\frac{m\pi}{L}\right)^2 \left(\frac{n\pi}{L}\right)^2 \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right)$$

Since $\sin\left(\frac{m\pi x}{L}\right)$ and $\sin\left(\frac{n\pi x}{L}\right)$ are orthogonal for $m \neq n$, we have

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = \begin{cases} 0 & m \neq n \\ L/2 & m = n \end{cases}$$

1

The total potential energy of the system is :

$$\Pi = U_i + U_e$$

$$= \frac{EI}{2} \frac{\pi^4}{2L^3} \sum_{n=1}^{\infty} n^4 a_n^2 + M_0 \left.\frac{dv}{dx}\right|_{x=c}$$

$$= \frac{EI}{2} \frac{\pi^4}{2L^3} \sum_{n=1}^{\infty} n^4 a_n^2 + M_0 \sum_{n=1}^{\infty} a_n \left(\frac{n\pi}{L}\right) \cos \frac{n\pi c}{L}$$

1

The variation of the total PE is zero about equilibrium:

$$\delta \Pi = \delta U_i + \delta U_e = 0$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{\partial U_i}{\partial a_n} \delta a_n - \sum_{n=1}^{\infty} \frac{\partial U_e}{\partial a_n} \delta a_n = 0$$

$$\Rightarrow \sum_{n=1}^{\infty} \left(\overbrace{\frac{\partial U_i}{\partial a_n} - \frac{\partial U_e}{\partial a_n}}^{\text{must be zero}} \right) \overset{\text{arbitrary}}{\delta a_n} = 0$$

1.5

$$\Rightarrow \frac{\partial U_i}{\partial a_n} = \frac{\partial U_e}{\partial a_n}$$

$$\Rightarrow \frac{\pi^4 EI}{2L^3} n^4 a_n = M_o \frac{n\pi}{L} \cos\left(\frac{n\pi c}{L}\right)$$

1.5

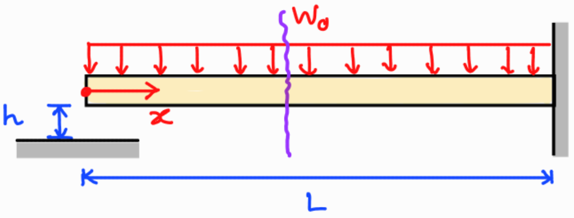
$$\Rightarrow a_n = \frac{2 M_o L^2}{\pi^3 EI} \frac{1}{n^3} \cos\left(\frac{n\pi c}{L}\right)$$

Upon substituting back the value of a_n in the series, we get:

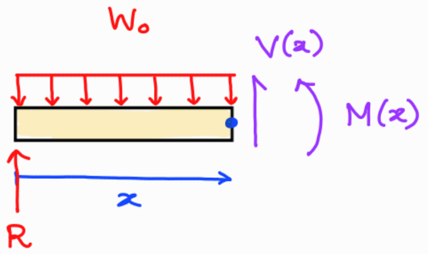
$$v(x) = \frac{2 M_o L^2}{\pi^3 EI} \sum_{n=1}^{\infty} \frac{1}{n^3} \cos\left(\frac{n\pi c}{L}\right) \sin\left(\frac{n\pi x}{L}\right)$$

1

Q2) a)



When the gap h is closed a reaction force R exists at the left end



$\circlearrowleft + \sum M_{right} = 0$

$$\Rightarrow \quad M(x) = Rx - \frac{w_0 x^2}{2}$$ 1

Use Euler- Bernoulli beam equation

$$EI \frac{d^2 v}{dx^2} = M(x)$$

$$\Rightarrow EI \frac{d^2 v}{dx^2} = Rx - \frac{w_0 x^2}{2}$$

$$\Rightarrow EI \frac{dv}{dx} = \frac{Rx^2}{2} - \frac{w_0 x^3}{6} + c_1$$

$$\Rightarrow EI\, v(x) = \frac{Rx^3}{6} - \frac{w_0 x^4}{24} + c_1 x + c_2$$ 0.5

Boundary conditions

1.5

$$\left.\begin{aligned} v(0) &= -h \\ v(L) &= 0 \\ \frac{dv}{dx}(L) &= 0 \end{aligned}\right\}$$

yields a solution

$$c_1 = \frac{w_0 L^3}{6} - \frac{RL^2}{2}$$

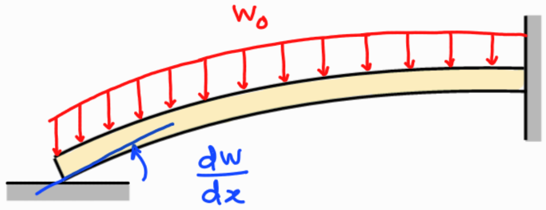
$$c_2 = -EIh$$

$$R = \frac{3}{8} w_0 L - \frac{3EIh}{L^3}$$ 1

The slope at the beam end

$$\frac{dv}{dx}(0) = \frac{c_1}{EI} = \frac{1}{EI}\left(\frac{w_0 L^3}{6} - \frac{3}{16} w_0 L^3 + \frac{3}{2}\frac{EIh}{L}\right) = \frac{3}{2}\frac{h}{L} - \frac{w_0 L^3}{48EI}$$ 0.5

b)



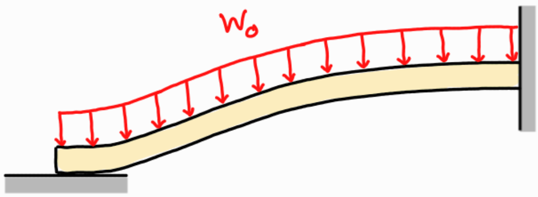
The first gap closes when $R \approx 0$

$$R = \frac{3}{8} w_0 L - \frac{3EIh}{L^3} = 0$$ 1

$$\Rightarrow \quad w_0 = \frac{8EIh}{L^4}$$ 0.5

Slope $$\frac{dv(0)}{dx} = \frac{c_1}{EI} = \frac{w_0 L^3}{6EI}$$ $[\because R = 0]$ 0.5

c) If we increase w_0, $\frac{dv(0)}{dx}$ decreases until it becomes 0



This occurs when,

$$\frac{dv(0)}{dx} = \frac{c_1}{EI} = 0 \quad \Rightarrow \quad c_1 = 0$$

$$\Rightarrow \frac{3}{2}\frac{h}{L} - \frac{w_0 L^3}{48EI} = 0$$

$$\Rightarrow w_0 = \frac{72EIh}{L^4}$$ 1

Corresponding reaction $$R = \frac{3}{8} w_0 L - \frac{3EIh}{L^3}$$

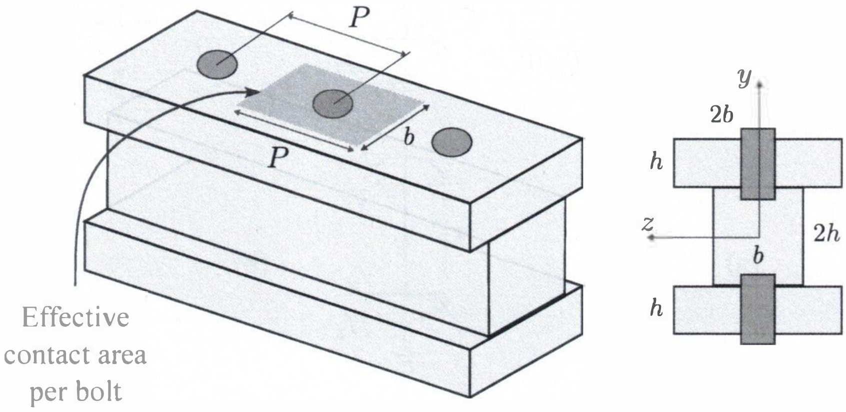
$$= \frac{3}{8}\left(\frac{72EIh}{L^4}\right) L - \frac{3EIh}{L^3}$$

$$= \frac{24EIh}{L^3}$$ 0.5

Q3)

The shear stress at the contact surface between two planks is given by:

$$\tau_{yx}(x,y) = \frac{V(x)Q(y)}{I_{zz}\, b(y)}$$

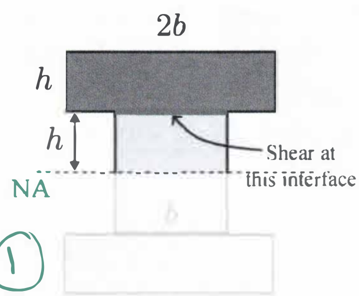


The effective area covered by each bolt has a length equal to the spacing between bolts. The total shearing force between the two planks must be resisted by the bolts. 1

$$\text{Resistive shear force per bolt} = \tau_{yx} \times \text{Effective contact area} \qquad 1$$
$$= \tau_{yx} \times (Pb)\ (b \text{ is taken as it is smaller out of the two width})$$

We need to find the shear stress at the interface of the two planks. Therefore, $Q(y = h)$ will only have a contribution from the shaded region.

$$Q(y=h) = \text{Moment of shaded area from NA}$$
$$= \text{Area of shaded region} \times \text{centroid of that area from NA} \qquad 1$$
$$= (2bh) \times \left(\frac{3h}{2}\right)$$
$$= 3bh^2 \qquad 2$$



V= F 1

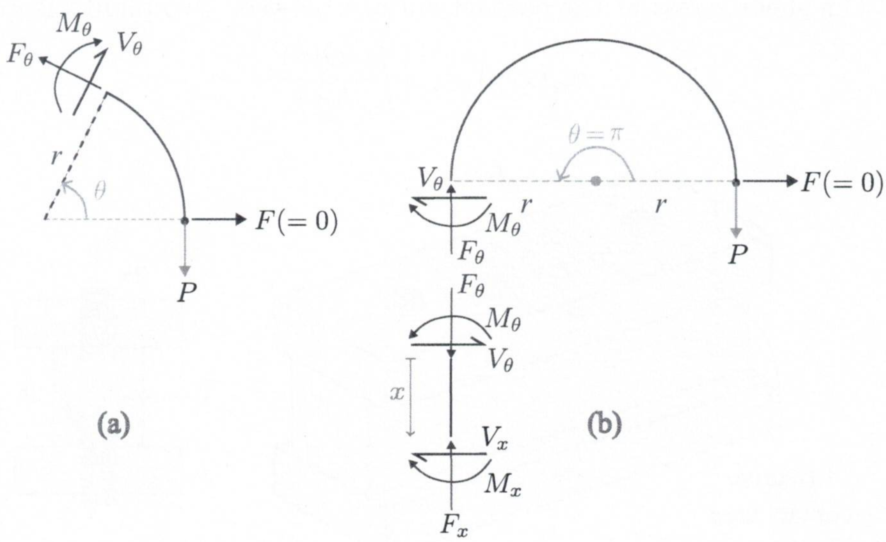
Assume that the shear force at any section is V, then

$$\tau_{yx} = \frac{V(3bh^2)}{10bh^3 \times b} \qquad 1$$
$$= \frac{3V}{10bh}$$
$$\text{Resistive shear force/bolt} = \frac{3V}{10bh} \times Pb$$
$$= \frac{3VP}{10h} \qquad 1$$

Solution 3: We are required to find both vertical and horizontal deflection of the beam at

Q4)

point A. We are given the corresponding force P at A to obtain the corresponding vertical deflection. However, there is no corresponding force for the horizontal displacement. Hence, we apply a dummy horizontal force $F(=0)$.



Next, we find the sectional forces and moments at an arbitrary cross-section. We take two cross-sections, one at an angle θ from A (see subfigure (a)) and another at a distance x (see subfigure (b)).

The procedure is as follows:
– First find out the internal shear force, axial force, bending moment, and twisting moment.
– Write down energy
– take derivative concerning the corresponding force to find the corresponding displacement.
In this problem, we neglect the energy due to shearing and stretching and only focus on energy due to bending. So we will only use the sectional bending moment for calculating the energy.

Bending moment at any section on the curved semi-circle:

$$\begin{aligned} &\sum M_o = 0 \\ \Rightarrow\ & -M_\theta - Pr\left(1-\cos\theta\right) + Fr\sin\theta = 0 \\ \Rightarrow\ & M_\theta = -Pr\left(1-\cos\theta\right) + Fr\sin\theta \end{aligned}$$

Bending moment at any section on the straight portion:

$$\begin{aligned} &\sum M_o = 0 \\ \Rightarrow\ & M_x - M_\theta\left.\right|_{\theta=\pi} + V_\theta\left.\right|_{\theta=\pi} x = 0 \\ \Rightarrow\ & M_x = -2Pr - Fx \end{aligned}$$

$$\begin{aligned} E\left(P,F\right) &= \int_0^\pi \frac{M_\theta^2}{2EI} r\,d\theta + \int_0^L \frac{M_x^2}{2EI} dx \\ &= \int_0^\pi \frac{\left(-Pr\left(1-\cos\theta\right)+Fr\sin\theta\right)^2}{2EI} r\,d\theta + \int_0^L \frac{\left(-2Pr - Fr\right)^2}{2EI} dx \end{aligned}$$

To obtain vertical deflection, δ_v, at point A, we take partial derivative of the energy w.r.t to the corresponding force P:

$$\begin{aligned}
\frac{\partial E}{\partial P}\Big|_{F=0} &= \int_0^{\pi} \frac{\left(-Pr\left(1-\cos\theta\right)+\cancelto{0}{F}x\sin\theta\right)}{EI}\left(-r(1-\cos\theta)\right)rd\theta + \int_0^L \frac{-2Pr-\cancelto{0}{F}r}{EI}(-2r)dx \\
&= \frac{Pr^3}{EI}\int_0^{\pi}\left(1-\cos\theta\right)^2 d\theta + \frac{4Pr^2L}{EI} \\
&= \frac{Pr^3}{EI}\left(\frac{3\theta}{2}-2\sin\theta+\frac{1}{4}\sin 2\theta\right)\Big|_0^{\pi} + \frac{4Pr^2L}{EI} \\
&= \frac{3\pi Pr^3}{2EI}+\frac{4Pr^2L}{EI} \\
&= \frac{Pr^2}{EI}\left[\frac{3\pi r}{2}+4L\right] \quad 3
\end{aligned}$$

To obtain horizontal deflection, δ_h, at point A, we take the partial derivative of the energy w.r.t to the corresponding dummy force F

$$\begin{aligned}
\frac{\partial E}{\partial F}\Big|_{F=0} &= \int_0^{\pi} \frac{\left(-Pr\left(1-\cos\theta\right)+\cancelto{0}{F}r\sin\theta\right)}{EI}\left(r\sin\theta\right)rd\theta + \int_0^L \frac{-2Pr-\cancelto{0}{F}x}{EI}(-x)dx \\
&= -\frac{Pr^3}{EI}\int_0^{\pi}\left(1-\cos\theta\right)\sin\theta d\theta + \frac{2Pr}{EI}\int_0^L x dx \\
&= -\frac{Pr^3}{EI}\left(\frac{\cos^2\theta}{2}-\cos\theta\right)\Big|_0^{\pi} + \frac{2Pr}{EI}\left(\frac{x^2}{2}\right)\Big|_0^L \\
&= -\frac{2Pr^3}{EI}+\frac{2PrL^2}{EI} \\
&= \frac{Pr}{EI}\left[-2r^2+2L^2\right] \quad 3
\end{aligned}$$

Equating δ_v to δ_h, we get

$$\begin{aligned}
\frac{Pr^2}{EI}\left[\frac{3\pi r}{2}+4L\right] &= \frac{Pr}{EI}\left[-2r^2+2L^2\right] \\
L^2-4Lr-r^2\left(\frac{3\pi}{2}+2\right) &= 0
\end{aligned}$$

Dividing by r^2 and putting $\dfrac{L}{r}=\rho$, we get

$$\rho^2-4\rho-\left(\frac{3\pi}{2}+2\right)=0$$

Upon solving the above, we get

$$\rho = \frac{4\pm\sqrt{16+4(3\pi/2+2)}}{2} = 2+\sqrt{6+\frac{3}{2}\pi}. \quad 4$$

Solution 4: Let the end reaction torques be T_L and T_R, respectively, at left and right fixed ends as shown in the figure below.