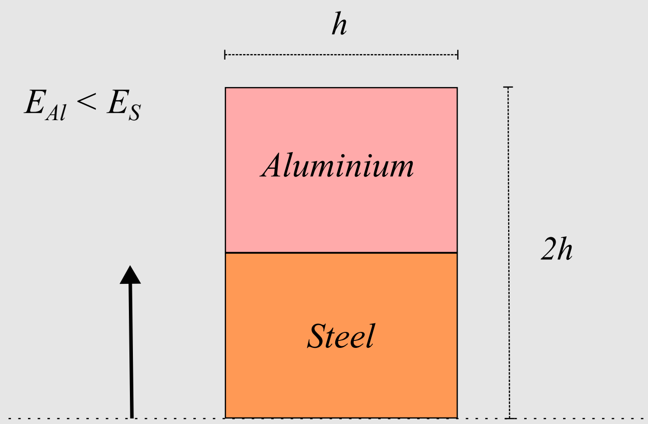
Tutorial 9 solution

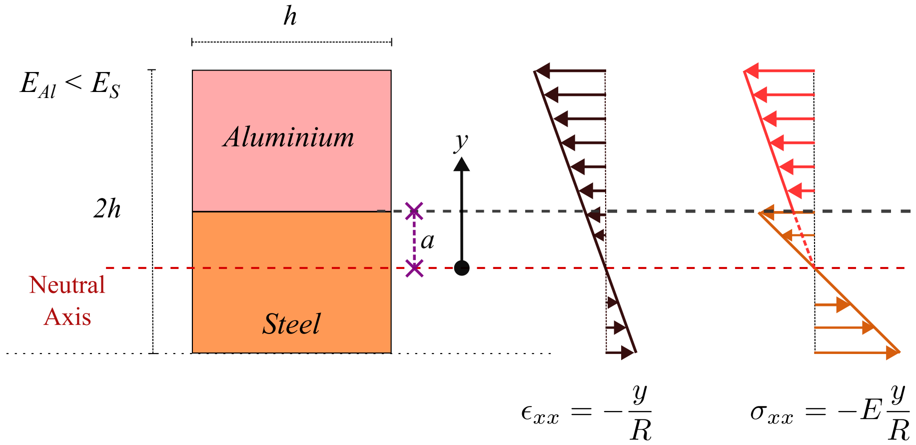
APL 104 - 2023 (Solid Mechanics)

Q1. Think of a composite beam having a rectangular cross-section such that one-half of the cross-section (having a square shape) is aluminium while the other half is steel. When such a beam is bent, where will the neutral axis lie in the cross-section (calculated from the bottom line of the cross-section)?



Solution: Composite beams are constructed from more than one material to increase their strength. In class, you have only seen beams made up of just one material. In general, the neutral axis of a composite beam will not lie at the geometric centroid of the cross-section.

Let us suppose that the neutral axis (NA) lies at a distance 'a' from the midline of the cross-section. The bending strain profile will be linear, continuous and passes through zero at the NA as shown below.



It obeys the following formula:

$$\epsilon_{xx} = -\frac{y}{R}$$

where y is the distance from the neutral axis. However, the bending stress profile will be discontinuous at the location where the material changes. It follows the folowing formula:

$$\sigma_{xx} = E\epsilon_{xx} = \begin{cases} \sigma_{xx}^S = -E_S \frac{y}{R} & (-(h-a) < y < a) \\ \sigma_{xx}^A = -E_A \frac{y}{R}, & (a < y < h+a) \end{cases}$$

To obtain 'a', we use the fact that the total axial force must vanish in the cross-section (due to pure bending), i.e.,

$$\begin{aligned} &\int\int_{A_S} \sigma_{xx}^S dA + \int\int_{A_A} \sigma_{xx}^A dA = 0 \\ \Rightarrow\ & -\frac{E_S}{R}\int_{-(h-a)}^{a} y dy - \frac{E_A}{R}\int_{a}^{h+a} y dy = 0 \\ \Rightarrow\ & \left[(a)^2 - (h-a)^2\right] + \frac{E_A}{E_S}\left[(h+a)^2 - (a)^2\right] = 0 \\ \Rightarrow\ & -h^2\left(1 - \frac{E_A}{E_S}\right) + 2ha\left(1 + \frac{E_A}{E_S}\right) = 0 \\ \Rightarrow\ & a = \frac{h}{2}\left(\frac{E_S - E_A}{E_S + E_A}\right). \end{aligned}$$

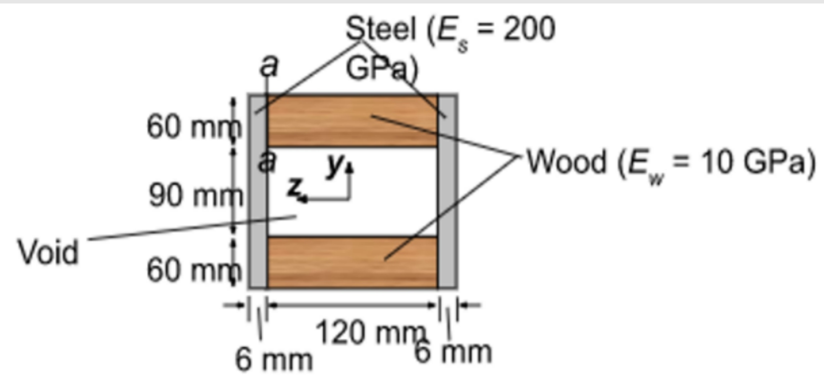
Note that when we set $E_s = E_A$ assuming both the materials to be the same, we indeed get $a = 0$ or the neutral axis then passes through the geometric center.

Q2. A beam of composite cross-section is subjected to bending moment $M_z = 30$kN. Find:

(a) The curvature $\kappa = \frac{1}{R}$ induced in the beam

(b) Maximum bending stress in wood

(c) Maximum bending stress in steel



Solution:
We first obtain the flexural rigidity of the cross-section. The cross-section being symmetrical,

the neutral axis will be the mid-horizontal line in the cross-section. The total flexural rigidity of the cross-section will simply be

$$
\begin{aligned}
(EI)_{tot} &= (EI)_{steel} + (EI)_{wood} \\
&= 2E_s \frac{1}{12} 0.006 \times 0.21^3 + E_w \frac{1}{12}\left(0.12 \times 0.21^3 - 0.12 \times 0.09^3\right).
\end{aligned}
$$

(i) The curvature induced in the beam would then simply be

$$\kappa = M_z/(EI)_{tot}$$

(ii) The maximum bending stress in the wood fiber will be in its topmost/bottom-most fiber, i.e.,

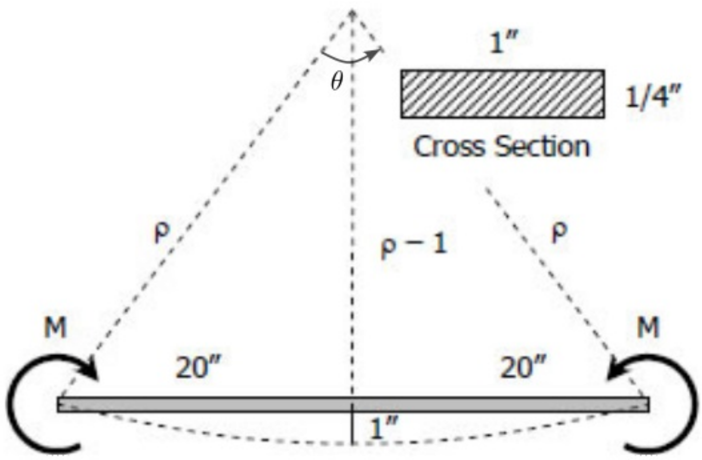
$$\sigma_w^{top} = E_w \kappa\ (0.045 + 0.060).$$

(iii) Likewise, the maximum bending stress in steel would be in its topmost/bottom-most fiber, i.e.,

$$\sigma_s^{top} = E_s \kappa\ (0.045 + 0.060).$$

Q3. A flat steel bar, 1 inch wide by 0.25 inch thick and 40 inch long, is bent by couples applied at the ends so that the midpoint deflection is 1 inch. Compute the stress in the bar and the magnitude of the applied couples. Use $E = 200$GPa.

Solution:



Note that the deflection of the beam is very small compared to its length. Therefore, in the above picture, we have drawn the ends of the deformed beam to coincide with the ends of the undeformed beam. In reality, the deformed beam and the undeformed beam will be of the same length due to pure bending deformation. However, with the deformed beam being curved, its two ends will be slightly inward compared to the undeformed beam. Neglecting this mismatch, we can write based on geometry that

$$
\begin{aligned}
&(\rho - \delta)^2 + L^2/4 = \rho^2 \\
&\Rightarrow \rho^2 - 2\rho\delta + \delta^2 + L^2/4 = \rho^2 \\
&\text{or } \rho = \frac{L^2 + 4\delta^2}{8\delta} = 200.5 \text{ in.}
\end{aligned}
$$

Another way to obtain the radius of curvature ρ is as follows. Let the angle subtended by the deformed beam at the center be $\theta/2$. So, $\rho\theta = L$. One can then write using trigonometry that

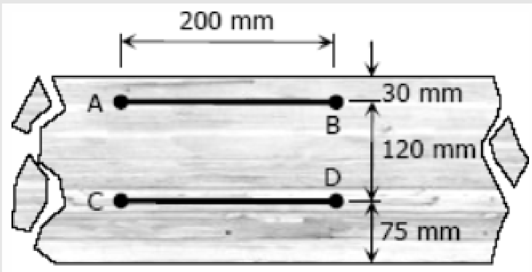
$$\begin{aligned}
&\rho\cos(\theta/2) = \rho - \delta \\
&\Rightarrow \rho(1 - \theta^2/8) = \rho - \delta \quad \text{(assuming } \theta \text{ to be very small)} \\
&\Rightarrow \rho\left(1 - \frac{L^2}{8\rho^2}\right) = \rho - 1 \\
&\text{or } \rho = \frac{L^2}{8\delta} = 200 \; in.
\end{aligned}$$

With the above value of ρ, θ turns out to be 0.2 radian or approximately 11 degrees. This validates our assumption. The couple required to generate this bending will be

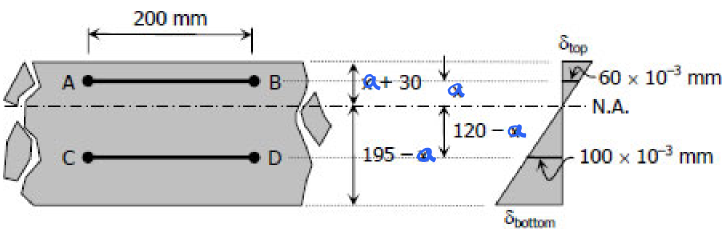
$$M = \frac{EI_{zz}}{\rho} = \frac{(29 \times 10^6)\,\frac{1\,(1/4)^3}{12}}{200.5} = 188.3 \text{ lb.in (answer)}.$$

Here, the Young's modulus E has been converted into the units of lb/in^2.

Q4. In a laboratory test of a beam loaded by end couples, the longitudinal fibers at layer AB in the figure below are found to increase 60×10^{-3}mm whereas those at CD decrease $100 \times 10^{-3} mm$ in the 200mm-gauge length. Using $E = 70GPa$, determine the flexural stress in the top and bottom fibers.



Solution: The picture above is that of the section of the beam along its length. The shape of the cross-section is not mentioned here. Accordingly, we can not assume that the neutral axis lies at the center. Let it lie at a distance a below the longitudinal fiber AB (see the figure below).



We know that $\epsilon_{xx} = -\frac{y}{R}$. Hence $\frac{y}{\epsilon_{xx}}$ must be a constant, i.e.,

$$\begin{aligned}
\frac{a}{\epsilon_{AB}} &= \frac{a-120}{\epsilon_{CD}} \\
\Rightarrow \frac{a}{\frac{60\times10^{-3}}{200}} &= \frac{a-120}{-\frac{100\times10^{-3}}{200}} \\
\Rightarrow a &= 0.6\,(120 - a) \\
\text{or } a &= 45\text{mm}.
\end{aligned}$$

We now obtain flexural/bending stress in the top fiber. We can write the following for the bending strain in the top fiber:

$$\frac{a}{\epsilon_{AB}} = \frac{a+30}{\epsilon_{top}} \Rightarrow \epsilon_{top} = \frac{a+30}{a}\epsilon_{AB} = \frac{75}{45}\frac{60\times10^{-3}}{200} = 5\times10^{-4}.$$

The bending stress in the top fiber will simply be $E\epsilon_{top}$ since the longitudinal fibers are under uniaxial loading during pure bending. Hence

$$\sigma_{top} = E\epsilon_{top} = 70\times10^{9} \ \times \ 5\times10^{-4} = 35\ MPa.$$

One can likewise obtain bending stress in the bottom-most fiber.