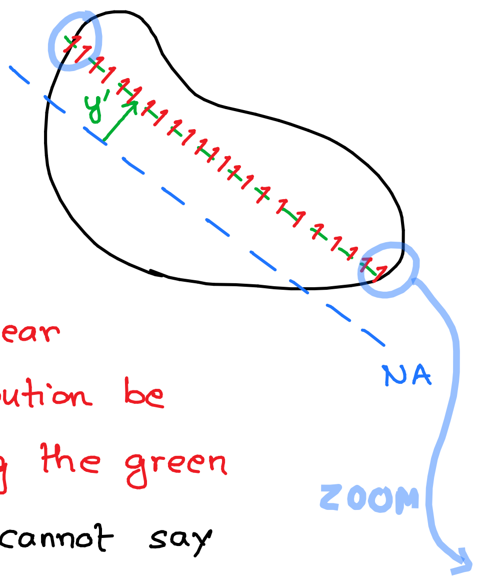
Non-uniform bending of unsymmetrical cross-sections

⇒ Shear force is present and bending moment varies across the length of the beam. The alignment of the neutral axis will be given by

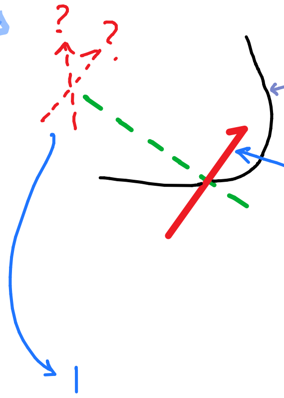
$$\frac{M_y}{M_z} = \frac{I_{yy}\cos\beta - I_{yz}\sin\beta}{I_{zz}\sin\beta - I_{yz}\cos\beta}$$

Now, to obtain the shear stress distribution in the C/s, let us look at a line lying at a distance y′ from the inclined NA



For symmetrical beam C/s, we had assumed that shear stress distribution does not vary along say the green line and our analysis got simpler. But can we go with the same assumption now?

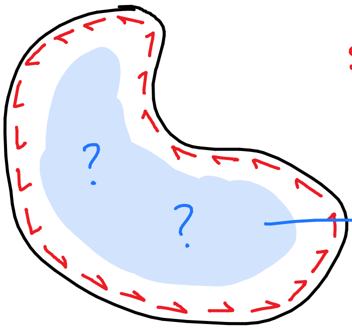
Will the shear stress distribution be uniform along the green line? We cannot say that for certain, because at the periphery of the C/s we should have a zero shear stress as the the sides of the beam are (shear) traction free. This fact is true for both symmetrical & unsymmetrical C/s



surface is traction-free

this stress component cannot exist near the periphery as it will then have a non-zero component along the peripheral surface normal leading to violation of traction-free surface

Hence, to obey the traction-free surface condition, be it a sym. or unsym. C/s, the shear stress near the periphery of the C/s must follow the peripheral outline.



Shear stress near the periphery aligns along the periphery of the C/s

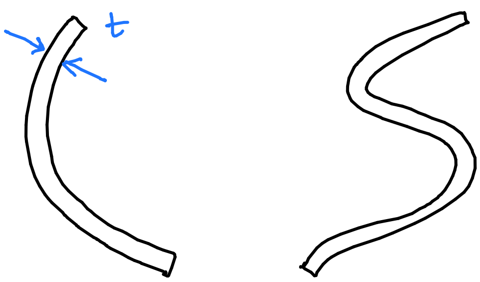
For inside (and away from the periphery, the shear stress could be directed in some way that cannot be easily predicted

Due to this complication, we won't get any analytical result of the variation of shear stress in such C/s. However, there are analytical results possible for thin and open cross-sections.

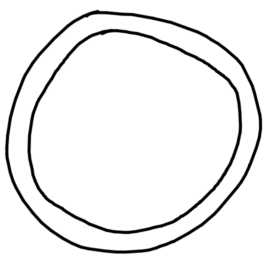
Shear stress distribution in thin and open cross-section

Thin → Thickness (or width) of beam C/s is small

Open → There are no closed loops formed in the C/s

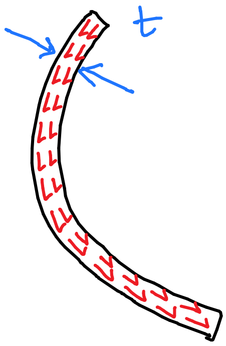


Thin & open C/s



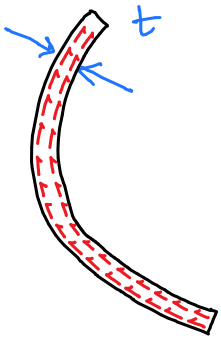
Thin & closed C/s

We already mentioned that shear stress at points close to the periphery must be oriented along the periphery.



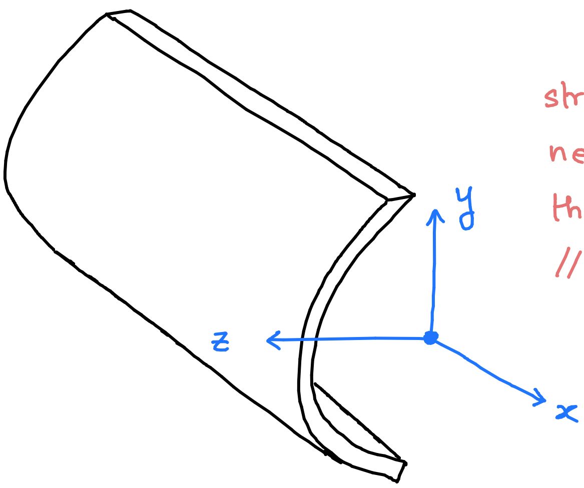
Thin $\Rightarrow$ shear stress is constant along thickness direction as the material points are all close to the periphery. Further, the shear stresses are all aligned along the periphery, so direction of flow of τ is known

OR

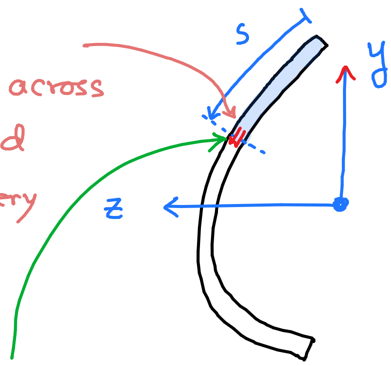


Open $\Rightarrow$ Shear stress flows from one end to other end of the c/s in a unidirectional fashion. In closed section, shear stress will not flow in one direction.

For such thin and open c/s, we can derive an analytical result of the variation of shear stress in the c/s. To do the analysis, we will use a slightly different coordinate system, an arc-coordinate system.



We expect the shear stresses will be nearly uniform across the thickness and // to the periphery

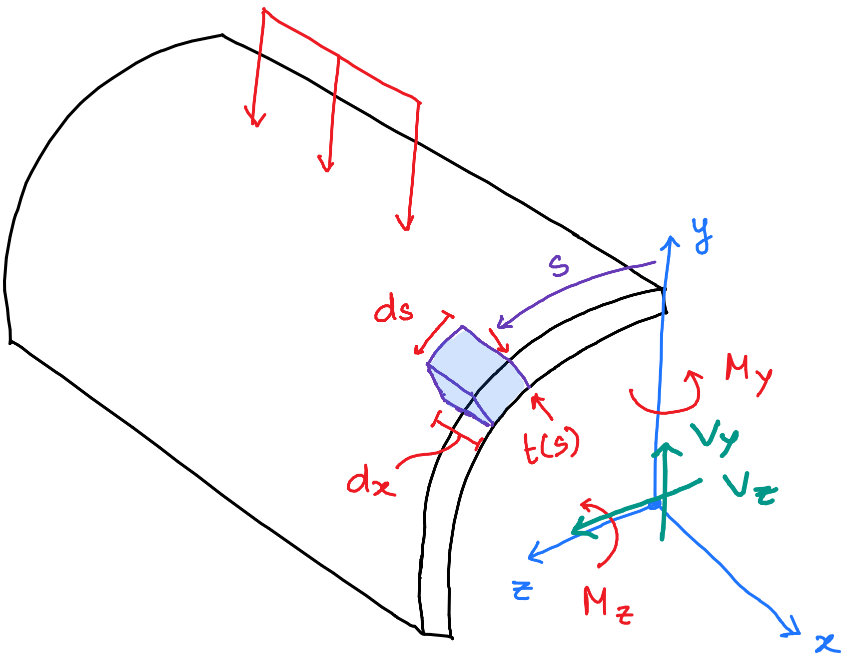


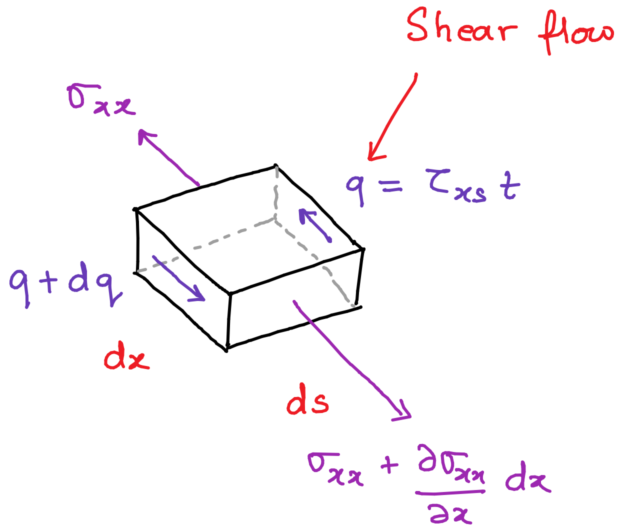
$\tau_{xs}(s)$

plane on which shear stress is acting

direction of shear stress

To find τ_{xs}, we consider a small part of the beam





The small element shows the changing bending stress and shear flow

Define shear flow,

$$q(s) = \tau_{xs}(s)\, t(s)$$

If we now sum the forces in the x-direction for this small element, we find

$$\xrightarrow{+} \sum F_x = 0$$

$$\Rightarrow (\cancel{q} + dq)\, dx - \cancel{q\, dx} + \left(\cancel{\sigma_{xx}} + \frac{d\sigma_{xx}}{dx} dx\right) dA - \cancel{\sigma_{xx}\, dA} = 0$$

$$\Rightarrow \quad dq = -\frac{d\sigma_{xx}}{dx} dA$$

If we now integrate the above equation between s = 0 to s = s,

$$\Delta q = q(s) - \cancel{q(0)} = \tau_{xs}\, t$$

0 as free end has no shear

Therefore,

$$\tau_{xs}\, t = -\int_{A'} \frac{d\sigma_{xx}}{dx}\, dA$$

where A' is the area of c/s between s = 0 & s = s

For an unsymmetrical c/s, the bending stress is given by

$$\sigma_{xx} = \frac{M_z\,(y\,I_{yy} - z\,I_{yz}) + M_y\,(y\,I_{yz} - z\,I_{zz})}{I_{yz}^2 - I_{yy}\,I_{zz}}$$

If we differentiate this bending stress, we can use the moment-shear relations

$$\frac{dM_z}{dx} = -V_y \,, \qquad \frac{dM_y}{dx} = V_z$$

So we get the expression for bending stress σ_{xx} as:

$$\frac{d\sigma_{xx}}{dx} = \frac{-V_y\,(y\,I_{yy} - z\,I_{yz}) + V_z\,(y\,I_{yz} - z\,I_{zz})}{I_{yz}^2 - I_{yy}\,I_{zz}}$$

$$\tau_{xs}\,t = -\int_{A'} \frac{d\sigma_{xx}}{dx}\,dA$$

constants for c/s

$$= -\int_{A'} \frac{-V_y\,(y\,I_{yy} - z\,I_{yz}) + V_z\,(y\,I_{yz} - z\,I_{zz})}{I_{yz}^2 - I_{yy}\,I_{zz}}\,dA$$

We only need to work out the integrals $\iint y\,dA$ and $\iint z\,dA$

Define $\displaystyle Q_y^s = \iint_{A'} y\,dA = \bar{y}^s A'$

Area of shaded region

$$Q_z^s = \iint_{A'} z\,dA = \bar{z}^s A'$$

$(\bar{y}^s, \bar{z}^s) \rightarrow$ coordinates of the centroid of the shaded area A'

Upon plugging the values of Q_y^s and Q_z^s back in the equation,

$$\boxed{\tau_{xs} = -\frac{\left[-V_y\left(Q_y^s I_{yy} - Q_z^s I_{yz}\right) + V_z\left(Q_y^s I_{yz} - Q_z^s I_{zz}\right)\right]}{t(s)\left(I_{yz}^2 - I_{yy} I_{zz}\right)}}$$

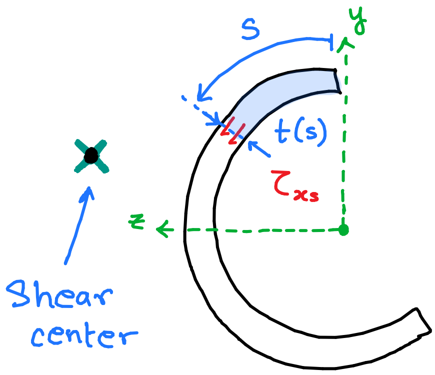
This is a general formula for shear stress distribution in thin & open cross-sections.

Special case: 'y' and 'z' axes align with principal axes of c/s

$I_{yz} = 0$!

$$\tau_{xs} = \frac{\left[-V_y Q_y^s I_{yy} - V_z Q_z^s I_{zz}\right]}{t(s)\, I_{yy} I_{zz}} = -\frac{V_y Q_y^s}{I_{zz}\, t(s)} - \frac{V_z Q_z^s}{I_{yy}\, t(s)}$$

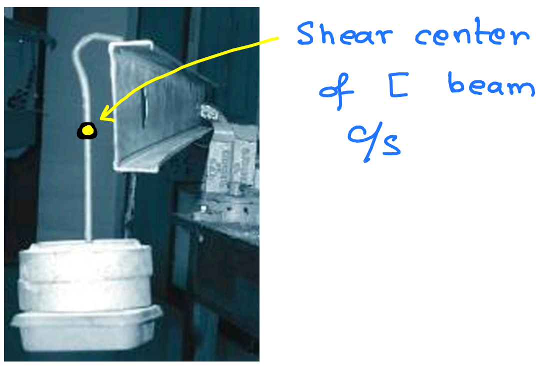
Concept of shear center



There is a point in the plane of the c/s about which the net torque due to shear stress distribution (arising due to transverse load) is zero

If a vertical force is applied through the shear center, then the beam will only bend. Else, the beam may bend as well as twist.

Bending & Twisting



Only bending