Non-uniform bending

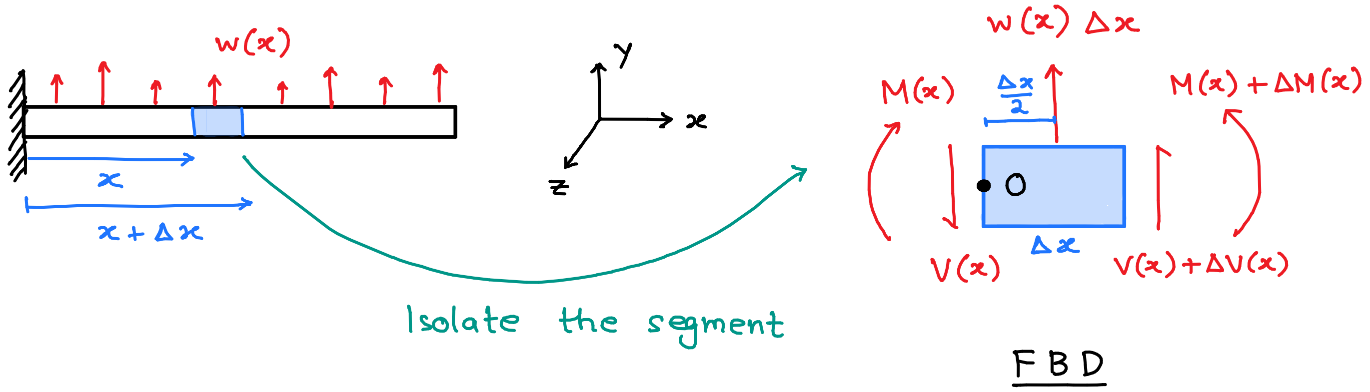
The case of pure bending leads to a constant curvature of the longitudinal fibers of the beam $\longrightarrow$ hence it is also called uniform bending.

When beams are used in practice, they normally support transverse loads rather than just applied bending moment (as was considered in pure bending). Physically, this has two consequences.

(1) The internal resistive bending moment will no longer be constant throughout the length of the beam. The varying internal moment in the beam can be found by applying equilibrium to an arbitrary section of the beam.

(2) The curvature of the longitudinal fibers will no longer remain constant but will vary throughout the length of the beam.

To understand these implications, let us consider a case of transverse loading, where we have a distributed load $w(x)$ acting on the beam.



Under equilibrium, net moment of the small portion must be zero:

$$\circlearrowleft + \sum M_O = 0 \Rightarrow -M + (V + \Delta V)\,\Delta x + w\,\Delta x\left(\frac{\Delta x}{2}\right) + M + \Delta M = 0$$

Origin at left c/s

$$\Rightarrow \quad -\cancel{M} + (V + \Delta V)\,\Delta x + w\,\frac{(\Delta x)^2}{2} + \cancel{M} + \Delta M = 0$$

$$\Rightarrow \quad V\,\Delta x + \underbrace{\Delta V\,\Delta x}_{} + \underbrace{w\,\frac{(\Delta x)^2}{2}}_{} + \Delta M = 0$$

product of small terms are ignored

$$\Rightarrow \quad V\,\Delta x + \Delta M = 0 \quad \Rightarrow \quad \frac{\Delta M}{\Delta x} = -V$$

Dividing by Δx and taking the limit as $\Delta x \to 0$, we get

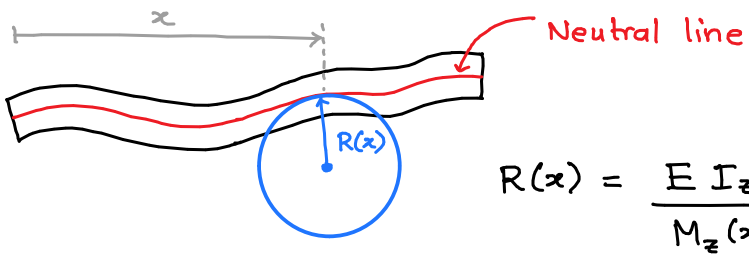
$$\boxed{\frac{dM(x)}{dx} = -V(x)}$$

$\leftarrow$ This relation captures the variation of bending moment and shear force

The above relation signifies that whenever bending moment varies along the beam, there has to be a non-zero shear force acting on the beam's cross-section.

So will a beam with constant and symmetrical cross-section bend into a perfect circle when there is transverse loads ?

NO. However, the beam will bend locally into an arc of a circle



$$R(x) = \frac{E\,I_{zz}}{M_z(x)}$$

The radius of curvature of the local circle can be obtained from the bending moment at the c/s as if it was in pure bending.

The corresponding bending stress at a cross-section located distance x from the origin is given by :

distance from neutral axis

$$\boxed{\sigma_{xx}(x, y, z) = -\frac{M(x)\,y}{I_{zz}}}$$

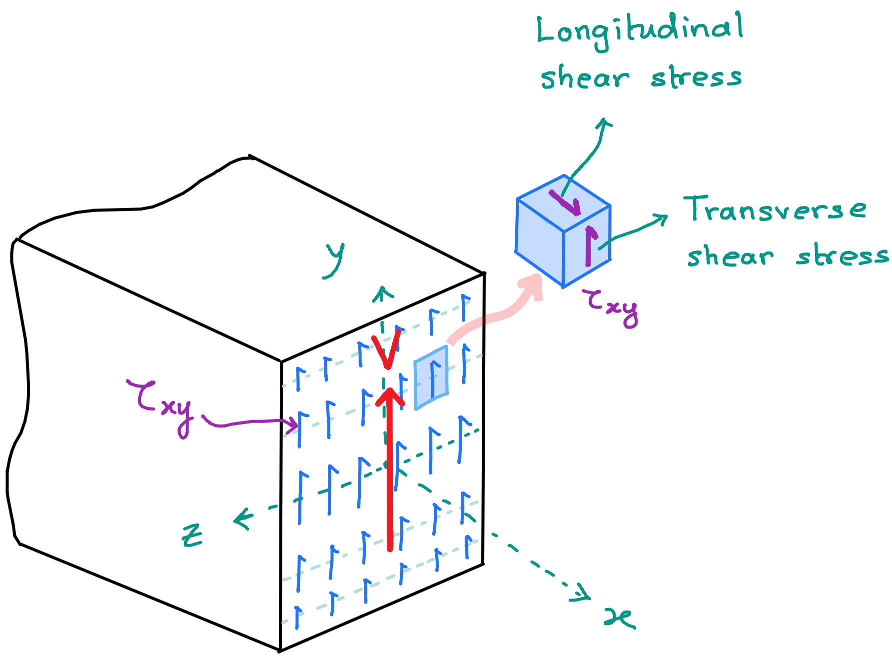
Distribution of shear stress τ_{xy} in the cross-section

The internal shear force $V(x)$ acting on a cross-section in the y-direction is the result of a transverse shear stress distribution τ_{xy} that acts over the beam's cross-section.

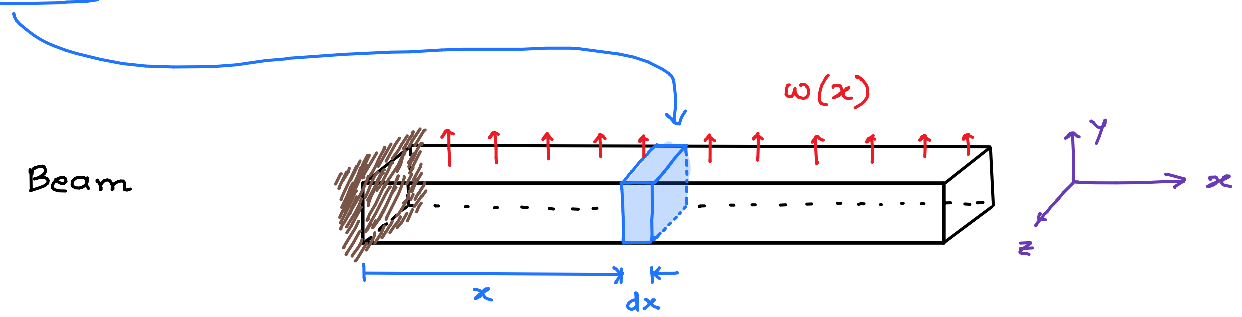
Here, we make a simplifying assumption that at a given cross-section

$$\boxed{\tau_{xy} \text{ is a function of } y \text{ only and not } z}$$

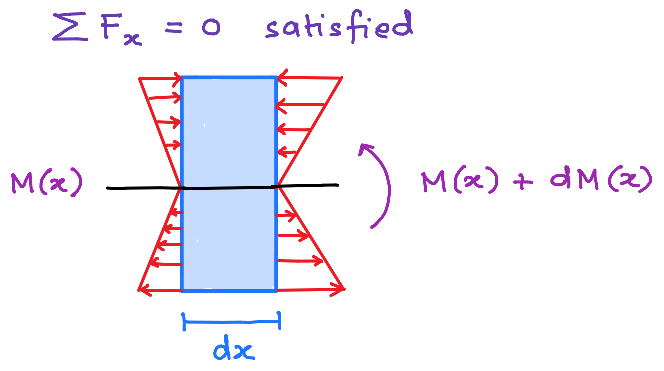
This means that τ_{xy} would remain the same at all points on lines parallel to the z-axis; different horizontal lines will have different τ_{xy} though.



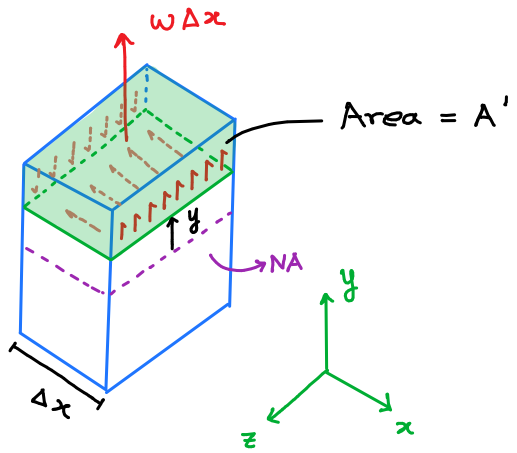
To obtain the shear-stress distribution on the cross-section, we consider the horizontal force equilibrium of a portion of an element taken from the beam shown below!



(a) Beam

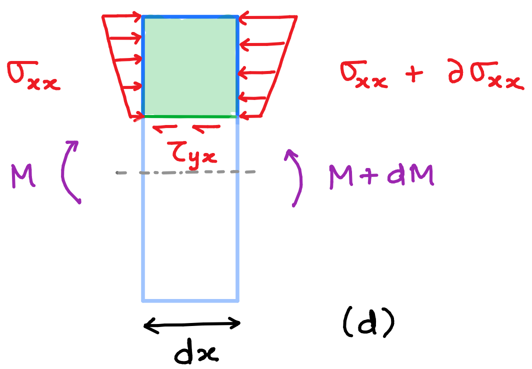


(b) FBD showing horizontal forces

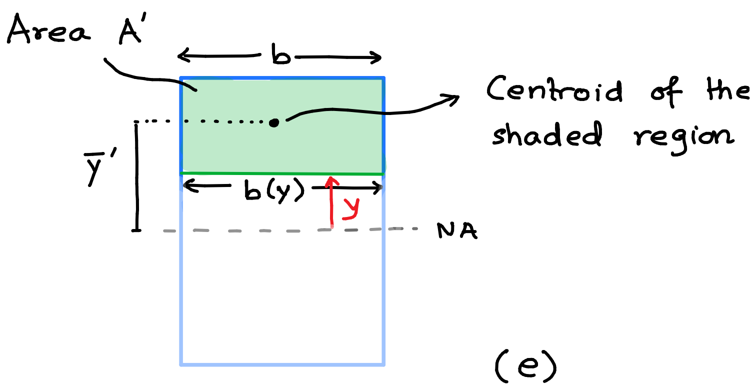


(c) Shaded region

Now consider the shaded top portion of the element that has been sectioned at y from the neutral axis (Fig (c)). The shaded portion has a width $b(y)$ $(= b)$ at a distance y from NA



(d)



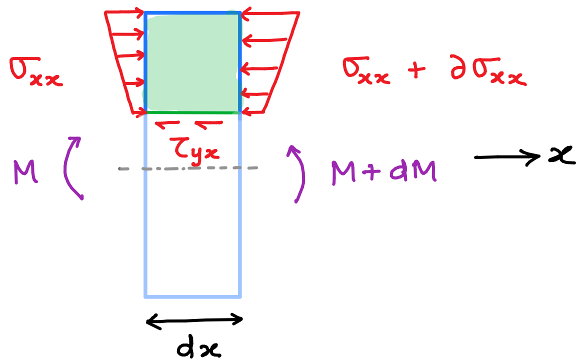
(e)

We only show the horizontal forces and moments in Fig (d)

- Left face (−ve x-plane) : Bending stress σ_{xx}, (also transverse τ_{xy})
- Right face (+ve x-plane) : Bending stress $\sigma_{xx} + \partial\sigma_{xx}$ (also transverse $\tau_{xy} + \partial\tau_{xy}$)
- Bottom face (−ve y-plane) : Longitudinal shear stress τ_{yx} $(= \tau_{xy})$

- Top face (+ve y-plane) : Transverse $w(x)$ acts
- Side faces (+ve & -ve z-planes) : Traction-free

We will now balance the horizontal forces acting on the shaded element in the x-direction:



$$\xrightarrow{+} \sum F_x = 0$$

$$\Rightarrow \int_{A'} \sigma_{xx}\, dA' - \int_{A'} (\sigma_{xx} + \partial\sigma_{xx})\, dA' - \tau_{yx}\, b(y)\, dx = 0$$

Use $\sigma_{xx} = \dfrac{M(x)\, y}{I_{zz}}$

$$\Rightarrow \int_{A'} \left(\frac{M}{I_{zz}}\right) y\, dA' - \int_{A'} \left(\frac{M+dM}{I_{zz}}\right) y\, dA' - \tau_{yx}\, b(y)\, dx = 0$$

$$\Rightarrow -\left(\frac{dM}{I_{zz}}\right) \int_{A'} y\, dA' = \tau_{yx}\, b(y)\, dx$$

$$\Rightarrow \tau_{yx} = \frac{1}{I_{zz}\, b(y)} \left(\overbrace{-\frac{dM(x)}{dx}}^{V(x)} \right) \underbrace{\int_{A'} y\, dA'}_{Q(y)}$$

This integral is the moment of area A' about the neutral axis, denoted by $Q(y)$

Formula for shear stress distribution

$$\boxed{\tau_{xy} = \tau_{yx} = \frac{V(x)\, Q(y)}{I_{zz}\, b(y)}}$$

$$Q(y) = \int_{A'} y\, dA' = \underbrace{\bar{y}'}_{\text{location of centroid of } A' \text{ from NA}} A'$$

Shear formula

Internal shear force determined from equilibrium equations

Shear stress at a point located a distance y from the NA. The stress is assumed constant over the width at a given y.

$$\tau_{xy} = \frac{V(x)\,Q(y)}{I_{zz}\,b(y)}$$

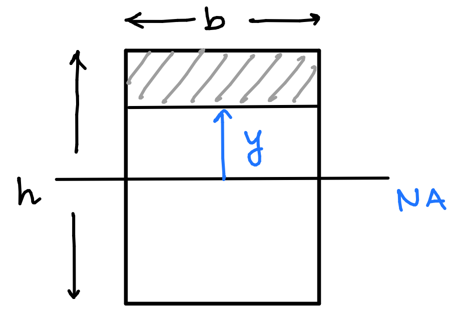
$= \bar{y}'A'$, where A' is the area of the top shaded portion of the entire C/s area and $\bar{y}'$ is the distance of centroid of A' from NA

Moment of inertia of the entire C/s area about NA

breadth (or width) of member's cross-sectional area measured at the point where τ_{xy} is to be determined

Although the shear formula was derived for finding the longitudinal stresses, it was used for transverse shear stress distribution, since τ_{yx} and τ_{xy} are complimentary and numerically equal.

<u>Rectangular C/S</u>

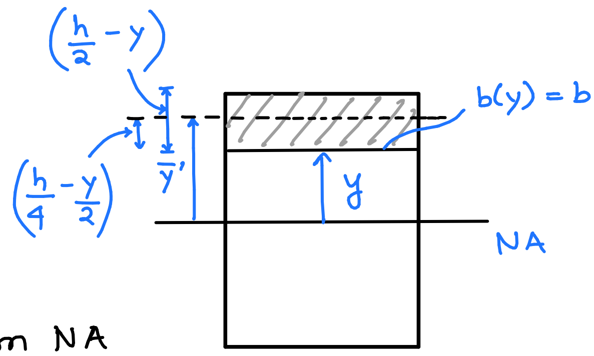


We want to find the value of τ_{xy} at a distance of y from NA

$Q(y)$, $b(y)$, I_{zz} ?

- $I_{zz} = \frac{1}{12} b h^3$

- Determine $Q(y)$ for shaded region



Centroid $\bar{y}'$ of the shaded region

$$\bar{y}' = y + \frac{1}{2}\left(\frac{h}{2} - y\right) = \frac{1}{2}\left(y + \frac{h}{2}\right) \quad \text{from NA}$$

Area of the shaded region

$$A' = b\left(\frac{h}{2} - y\right)$$

$$Q = \bar{y}' A' = \frac{1}{2}\left(y + \frac{h}{2}\right) b\left(\frac{h}{2} - y\right) = \frac{1}{2} b \left(\frac{h^2}{4} - y^2\right)$$

So, the shear stress distribution becomes:

$$\tau_{xy} = \frac{V(x) \; \frac{1}{2} b \left(\frac{h^2}{4} - y^2\right)}{\frac{1}{12} b h^3 \; b} = \frac{V}{bh} \, 6\left(\frac{1}{4} - \left(\frac{y}{h}\right)^2\right)$$