In the last lecture, using the physical considerations of axisymmetry, no warping/bulging of c/s, axial homogeneity, and torsion, we obtain simplied equilibrium eqns and displacement functions.

Solving the simplified equations

As u_r and u_z are functions of r and z, respectively, we can rewrite σ_{zz} as

$$\sigma_{zz} = \lambda\left(u_r' + \frac{u_r}{r} + u_z'\right) + 2\mu u_z'$$

$$= \underbrace{(\lambda + 2\mu)\, u_z'}_{\substack{\text{depends}\\ \text{upon } z}} + \underbrace{\lambda\left(u_r' + \frac{u_r}{r}\right)}_{\substack{\text{depends}\\ \text{upon } r}}$$

But from one of the equilibrium equations, we have that

$$\frac{\partial \sigma_{zz}}{\partial z} = 0 \quad \Rightarrow \quad \sigma_{zz} \text{ does not depend upon } z$$

$$\Rightarrow \quad \sigma_{zz} = f(r) \quad \Big\} \text{ must be a function of } r \text{ only}$$

$\therefore$ The term u_z' must be a constant

$$\Rightarrow u_z = c_1 z + c_2$$

Also, $u_z = 0$ at the cylinder's mid-section $(z=0)$ [disp BC]

Therefore, $c_2 = 0 \quad \Rightarrow \quad u_z = c_1 z \leftarrow$ expression for u_z

Expression for u_r

We now look at the radial component of equilibrium equation

The expressions of σ_{rr} and $\sigma_{\theta\theta}$ become

$$\sigma_{rr} = \lambda\left(u_r' + \frac{u_r}{r} + u_z'\right) + 2\mu u_r' \qquad [\text{Use } u_z = c_1 z]$$

$$= \lambda\left(u_r' + \frac{u_r}{r} + c_1\right) + 2\mu u_r'$$

$$\sigma_{\theta\theta} = \lambda\left(u_r' + \frac{u_r}{r} + c_1\right) + 2\mu \frac{u_r}{r}$$

We will make use of the simplified equilibrium eqn,

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0$$

Subtracting $\sigma_{rr} - \sigma_{\theta\theta}$, we get

$$\sigma_{rr} - \sigma_{\theta\theta} = 2\mu\left(u_r' - \frac{u_r}{r}\right)$$

The partial derivative $\frac{\partial \sigma_{rr}}{\partial r}$ can be obtained as:

$$\frac{\partial \sigma_{rr}}{\partial r} = \lambda \frac{d}{dr}\left(u_r' + \frac{u_r}{r}\right) + 2\mu u_r'' \qquad [\because c_1 \text{ is constant}]$$

Plugging these values into the equilibrium eqn, we get

$$\lambda \frac{d}{dr}\left(\frac{du_r}{dr} + \frac{u_r}{r}\right) + 2\mu \frac{d^2 u_r}{dr^2} + \frac{2\mu}{r}\left(\frac{du_r}{dr} - \frac{u_r}{r}\right) = 0$$

$$\Rightarrow \lambda \frac{d}{dr}\left(\frac{du_r}{dr} + \frac{u_r}{r}\right) + 2\mu \frac{d}{dr}\left(\frac{du_r}{dr} + \frac{u_r}{r}\right) = 0$$

$$\Rightarrow \quad (\lambda + 2\mu)\left(\frac{d}{dr}\left(\frac{du_r}{dr} + \frac{u_r}{r}\right)\right) = 0$$

$$\Rightarrow \quad \frac{d}{dr}\left(\frac{du_r}{dr} + \frac{u_r}{r}\right) = 0$$

$$\Rightarrow \quad \underbrace{\frac{du_r}{dr}}_{\epsilon_{rr}} + \underbrace{\frac{u_r}{r}}_{\epsilon_{\theta\theta}} = \text{constant, say } C$$

The above expression also implies that $\epsilon_{rr} + \epsilon_{\theta\theta}$ = constant

To simplify a bit more, lets rewrite:

$$\frac{du_r}{dr} + \frac{u_r}{r} = C$$

$$\Rightarrow \quad \frac{1}{r}\frac{d}{dr}(r u_r) = C$$

$$\Rightarrow \quad r u_r = \frac{Cr^2}{2} + D$$

$$\Rightarrow \quad u_r = \frac{Cr}{2} + \frac{D}{r}$$

are unknown integration constants

These unknown integration constants C and D are obtained from boundary conditions.

If we substitute the expression of u_r and u_θ into the stress-strain relations, we will obtain expressions for radial stress σ_{rr} and hoop stress $\sigma_{\theta\theta}$.

Solution for radial stress σ_{rr} and hoop stress $\sigma_{\theta\theta}$

We find that, just like $\epsilon_{rr} + \epsilon_{\theta\theta} =$ constant, $\sigma_{rr} + \sigma_{\theta\theta}$ is also a constant

$$\sigma_{rr} + \sigma_{\theta\theta} = \lambda\left(u_r' + \frac{u_r}{r} + c_1\right) + 2\mu u_r' + \lambda\left(u_r' + \frac{u_r}{r} + c_1\right) + 2\mu\frac{u_r}{r}$$

$$= 2\lambda\left(u_r' + \frac{u_r}{r}\right) + 2\mu\left(u_r' + \frac{u_r}{r}\right) + 2\lambda c_1$$

$$= 2(\lambda+\mu)\underbrace{\left(u_r' + \frac{u_r}{r}\right)}_{\epsilon_{rr} + \epsilon_{\theta\theta}} + 2\lambda c_1$$

$$= 2(\lambda+\mu)C + 2\lambda c_1 = A \quad \text{(a constant)}$$

So the sum of radial stress and hoop stress turns out to be a constant throughout the hollow cylinder

If we now solve the simplified equilibrium equation directly in terms of stress components, we get:

[Note the partial deriv. has become total deriv. since σ_{rr} is function of r only]

$$\frac{d\sigma_{rr}}{dr} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0$$

$$\Rightarrow \frac{d\sigma_{rr}}{dr} + \frac{2\sigma_{rr}}{r} - \frac{\overbrace{\sigma_{rr} + \sigma_{\theta\theta}}^{\text{constant } A}}{r} = 0$$

$$\Rightarrow \frac{d\sigma_{rr}}{dr} + \frac{2\sigma_{rr}}{r} = \frac{A}{r}$$

$$\Rightarrow \frac{1}{r^2}\frac{d}{dr}\left(\sigma_{rr} r^2\right) = \frac{A}{r}$$

$$\Rightarrow \sigma_{rr} = \frac{A}{2} + \frac{B}{r^2} \quad \text{and} \quad \sigma_{\theta\theta} = A - \sigma_{rr} = \frac{A}{2} - \frac{B}{r^2}$$

*A and B are unknown integration constants.

Use of boundary conditions for finding unknown integration constants

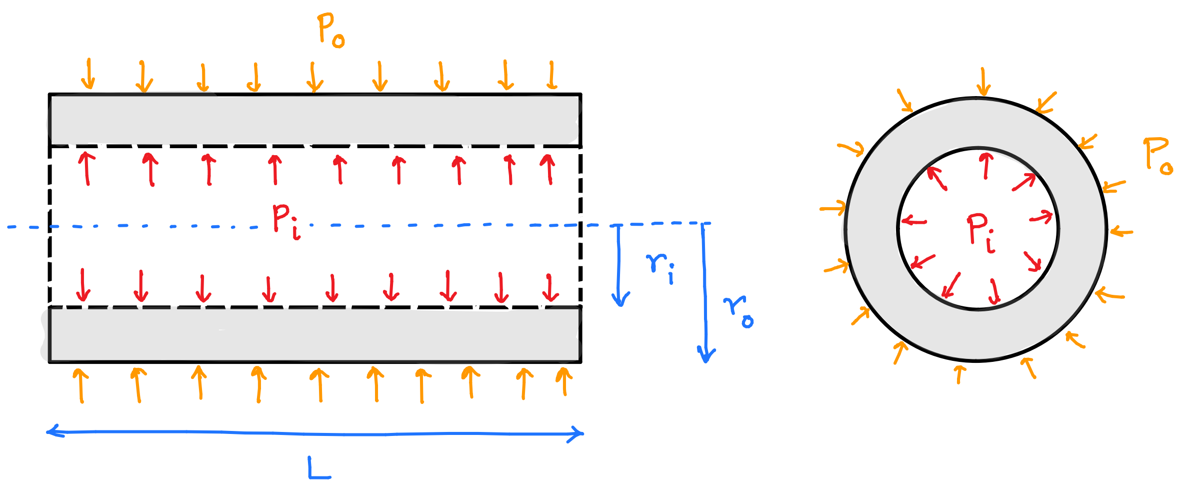
If we now satisfy boundary conditions, then we will have "unique" elasticity solutions

(a) External and internal pressure loading

Cylindrical vessel is loaded by uniform pressures (p_i, p_o) on the inside and outside surfaces of the cylinder.

We assume the ends of the cylinder are traction-free

A real pressure vessel will have its ends closed in some fashion and we will deal with that case in axial loading.



For this case, the boundary conditions will be defined by the outward surface normal. The inner curve surface (at $r = r_i$), the outward surface normal points in the $-r$-direction

$$[\underline{n}]_{(r_i,\, \theta,\, z)} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

while the internal pressure p_i acts in the $+r$ direction

So, $$\left[t^{ext}\right]_{(r_i,\,\theta,\,z)} = \begin{bmatrix} P_i \\ 0 \\ 0 \end{bmatrix}$$

Upon writing the boundary condition equation $\underline{\underline{\sigma}}\,\underline{n} = \underline{t}^{ext}$ in cylindrical coordinate system for the inner surface, we get:

$$\begin{bmatrix} \sigma_{rr} & \tau_{r\theta} & \tau_{rz} \\ \tau_{r\theta} & \sigma_{\theta\theta} & \tau_{\theta z} \\ \tau_{rz} & \tau_{\theta z} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} P_i \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -\sigma_{rr} \\ -\tau_{r\theta} \\ -\tau_{rz} \end{bmatrix} = \begin{bmatrix} P_i \\ 0 \\ 0 \end{bmatrix}$$

or, $$\sigma_{rr}(r_i) = -P_i \,, \quad \tau_{r\theta} = \tau_{rz} = 0$$

A similar analysis for the outer curve surface would lead to

$$\sigma_{rr}(r_o) = -P_o \,, \quad \tau_{r\theta} = \tau_{rz} = 0$$

Effectively, we obtain the BCs

$$\sigma_{rr}(r_i) = -P_i$$

$$\sigma_{rr}(r_o) = -P_o$$

Plugging the above BCs in the expression for σ_{rr}, we get:

$$\frac{A}{2} + \frac{B}{r_i^2} = -P_i \,, \qquad \frac{A}{2} + \frac{B}{r_o^2} = -P_o$$

Solving for (A, B), we get

$$A = 2\,\frac{r_i^2 P_i - r_o^2 P_o}{r_o^2 - r_i^2} \,, \qquad B = \frac{r_i^2 r_o^2 (P_o - P_i)}{r_o^2 - r_i^2}$$

Therefore, the radial and hoop sl esses become

$$\sigma_{rr} = \frac{A}{2} + \frac{B}{r^2} = \frac{P_i r_i^2}{r_o^2 - r_i^2}\left(1 - \frac{r_o^2}{r^2}\right) - \frac{P_o r_o^2}{r_o^2 - r_i^2}\left(1 - \frac{r_i^2}{r^2}\right)$$

$$\sigma_{\theta\theta} = \frac{A}{2} - \frac{B}{r^2} = \frac{P_i r_i^2}{r_o^2 - r_i^2}\left(1 + \frac{r_o^2}{r^2}\right) - \frac{P_o r_o^2}{r_o^2 - r_i^2}\left(1 - \frac{r_i^2}{r^2}\right)$$

Note that both radial and hoop stresses vary with the radial coordinate r and depend linearly on the pressure loadings P_i & P_o. They are, however, independent of the axial loading.

To study the behavior of σ_{rr} and $\sigma_{\theta\theta}$, we consider several special cases.

(a) Inner and outer pressures are both equal ($P_i = P_o = p$)

$$\sigma_{rr} = \frac{p r_i^2}{r_o^2 - r_i^2}\left(1 - \frac{r_o^2}{r^2}\right) - \frac{p r_o^2}{r_o^2 - r_i^2}\left(1 - \frac{r_i^2}{r^2}\right)$$

$$= p \frac{1}{\cancel{r_o^2 - r_i^2}}\left[\cancel{r_i^2} - \cancel{\frac{r_i^2 r_o^2}{r^2}} - \cancel{r_o^2} + \cancel{\frac{r_i^2 r_o^2}{r^2}}\right]$$

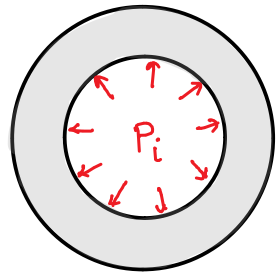
$$= -p$$

Similarly, $\sigma_{\theta\theta} = -p$

$$\therefore \quad \boxed{\sigma_{rr} = \sigma_{\theta\theta} = -p}$$

(b) When only internal pressure is present $(P_o = 0)$

$$\sigma_{rr} = \frac{P_i r_i^2}{r_o^2 - r_i^2}\left(1 - \frac{r_o^2}{r^2}\right)$$



At $r = r_i$, $\sigma_{rr} = -P$ (inner surface)

At $r = r_o$, $\sigma_{rr} = 0$ (outer surface)

The inner pressure results in a compressive radial stress which varies from $\sigma_{rr} = -P$ at inner wall to $\sigma_{rr} = 0$ at the outer wall

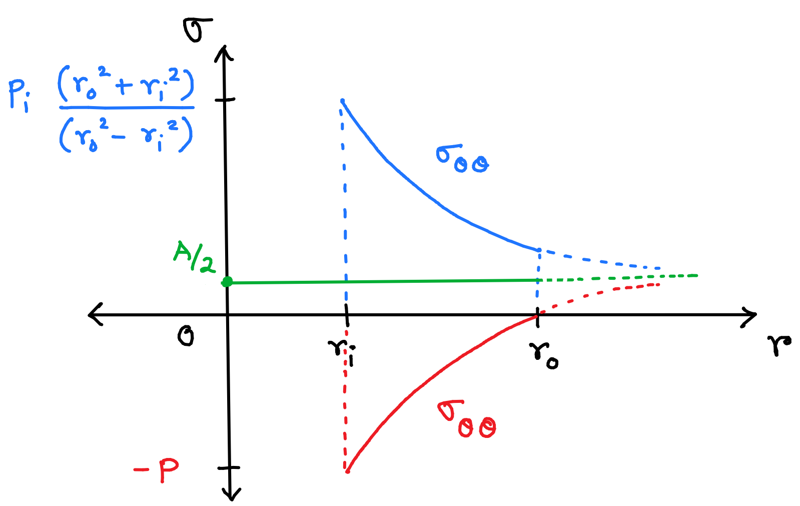
$$\sigma_{\theta\theta} = \frac{P_i r_i^2}{r_o^2 - r_i^2}\left(1 + \frac{r_o^2}{r^2}\right)$$

At $r = r_i$, $\sigma_{\theta\theta} = P_i \dfrac{(r_o^2 + r_i^2)}{(r_o^2 - r_i^2)}$

At $r = r_o$, $\sigma_{\theta\theta} = P_i \dfrac{2 r_i^2}{(r_o^2 - r_i^2)}$

The hoop stress $\sigma_{\theta\theta}$ is tensile throughout with peak magnitude occuring at the inner surface $r = r_i$. So the highest stresses for an internally pressurized cylinder occur on the inner wall.

The variation of σ_{rr} and $\sigma_{\theta\theta}$ with r is shown below:



* Keep in mind that the radial coordinate r is physically realizable only betn r_i and r_o

Final step to obtain u_r

The radial displacement u_r was obtained as: $u_r = \frac{Cr}{2} + \frac{D}{r}$

To obtain the constants (C, D), we can make use of the stress-strain relation

$$\sigma_{rr} = \lambda\left(\underbrace{\underbrace{u_r'}_{\epsilon_{rr}} + \underbrace{\frac{u_r}{r}}_{\epsilon_{\theta\theta}}}_{C} + c_1\right) + 2\mu u_r'$$

$$= \lambda\left(C + c_1\right) + 2\mu\left(\frac{C}{2} - \frac{D}{r^2}\right)$$

$$= \left[(\lambda+\mu)\, C + \lambda c_1\right] - 2\mu \frac{D}{r^2}$$

Comparing the above equation with expression obtained for σ_{rr}

$$\sigma_{rr} = \frac{A}{2} + \frac{B}{r^2}$$

$$\left[(\lambda+\mu)\, C + \lambda c_1\right] = \frac{A}{2} = \frac{r_i^2 P_i - r_o^2 P_o}{r_o^2 - r_i^2}$$

① $$C = -\frac{\lambda}{\lambda+\mu} c_1 + \frac{1}{\lambda+\mu} \frac{P_i r_i^2 - P_o r_o^2}{r_o^2 - r_i^2}$$

② $$D = -\frac{B}{2\mu} = -\frac{1}{2\mu} \frac{r_i^2 r_o^2 (P_o - P_i)}{r_o^2 - r_i^2}$$

Therefore, radial displacement u_r becomes

$$u_r = \frac{1}{2}\left(\frac{\lambda}{(\lambda+\mu)} c_1 + \frac{1}{(\lambda+\mu)} \frac{P_i r_i^2 - P_o r_o^2}{r_o^2 - r_i^2}\right) r - \frac{1}{2\mu r} \frac{r_i^2 r_o^2 (P_o - P_i)}{r_o^2 - r_i^2}$$

When both internal and external pressures are zero, i.e. $p_i = p_o = 0$, we get

$$u_r = \frac{-\lambda}{2(\lambda+\mu)} c_1 r = -\nu c_1 r \qquad \left(\because \nu = \frac{\lambda}{2(\lambda+\mu)}\right)$$

We see that radial strain is not zero but it turns out as

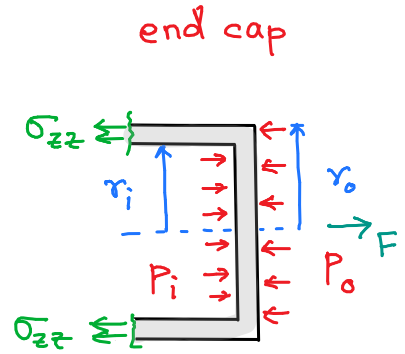
$$\epsilon_{rr} = \frac{du_r}{dr} = -\nu c_1$$

This expression is exactly what we expect, that is, radial displacement has arisen only due to Poisson's effect in the absence of radial pressure loading.

We now need to obtain axial strain ϵ_{zz} and the end-to-end rotation Ω in terms of prescribed external loads (axial force F, torque T, and pressures p_i and p_o)

Relating axial force and axial strain

Typically, the ends of pressure vessels are capped. In such a case the end caps are



(a) pushed out by internal pressure p_i

(b) pushed in by external pressure p_o

(c) pulled out by external force F

(d) balanced by internal traction due to σ_{zz}

Upon doing force balance, we can write

$$F + P_i \pi r_i^2 - P_o \pi r_o^2 = \iint_{A_o} \sigma_{zz} \, dA$$

$$\Rightarrow \quad F = \iint_{A_o} \left[\lambda \left(\underbrace{u_r' + \frac{u_r}{r}}_{C} + \underbrace{u_z'}_{c_1} \right) + 2\mu \underbrace{u_z'}_{c_1} \right] dA - P_i \pi r_i^2 + P_o \pi r_o^2$$

$$= \iint_{A_o} \Big[\underbrace{\lambda C + (\lambda + 2\mu) c_1}_{\text{constant}} \Big] dA - P_i \pi r_i^2 + P_o \pi r_o^2$$

$$= \left[\lambda C + (\lambda + 2\mu) c_1 \right] \iint_{A_o} dA - P_i \pi r_i^2 + P_o \pi r_o^2$$

$$= \left[\lambda C + (\lambda + 2\mu) c_1 \right] \pi (r_o^2 - r_i^2) - P_i \pi r_i^2 + P_o \pi r_o^2$$

Substitute the obtain expression for C

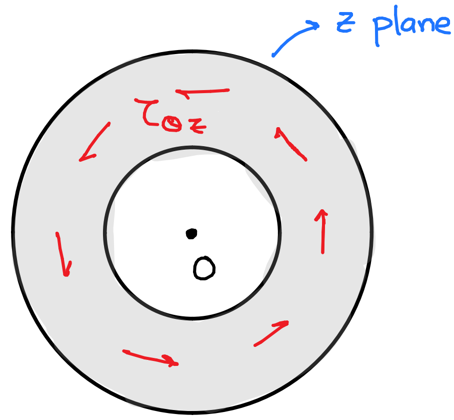
$$C = -\frac{\lambda}{\lambda + \mu} c_1 + \frac{1}{\lambda + \mu} \frac{P_i r_i^2 - P_o r_o^2}{r_o^2 - r_i^2}$$

$$F = \left[(\lambda + 2\mu) - \frac{\lambda^2}{\lambda + \mu} \right] \underbrace{\pi (r_o^2 - r_i^2)}_{C/s \text{ area}} c_1 - P_i \pi r_i^2 \left[1 - \frac{\lambda}{\lambda + \mu} \frac{1}{\pi (r_o^2 - r_i^2)} \right]$$

$$+ P_o \pi r_o^2 \left[1 - \frac{\lambda}{\lambda + \mu} \frac{1}{\underbrace{\pi (r_o^2 - r_i^2)}_{C/s \text{ area}}} \right]$$

Relating torque and end-to-end rotation

A typical C/s of the hollow cylinder would have its cross-sectional normal point along the z-direction. Therefore, stress components σ_{zz}, $\tau_{z\theta}$, τ_{zr} act on it.



We had already mentioned that

$$\tau_{zr} = 0$$

The moment M about the C/s's centroid can be calculated as:

$$\underline{M} = \iint_{A_0} r\underline{e}_r \times \underline{t}^z(r) \, dA$$

cross-product

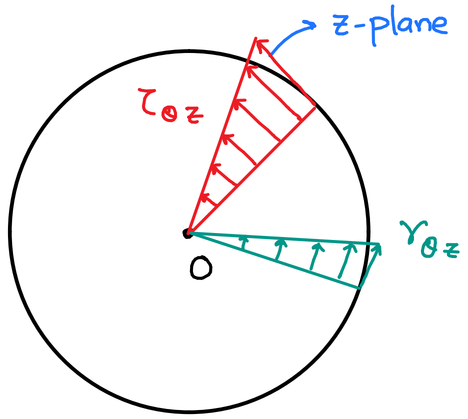
$$= \iint_{A_0} r\underline{e}_r \times \left[\sigma_{zz}\, \underline{e}_z + \tau_{\theta z}\, \underline{e}_\theta\right] dA$$

Before we obtain torque from M let's look at the variation of $\tau_{\theta z}$ in the cross-section. This $\tau_{\theta z}$ occurs as a result of applying torque.

We know, $\quad \gamma_{\theta z} = \dfrac{\Omega}{L} r \quad \Rightarrow \quad \tau_{\theta z} = G\gamma_{\theta z}$

$$= \frac{G\Omega}{L} r$$

$\Rightarrow$ $\tau_{\theta z}$ varies linearly with r



The torque T is simply the component of moment along the z-axis, i.e

$$T = \underline{M} \cdot \underline{e}_z = \iint_{A_o} \left(r \underline{e}_r \times \left[\sigma_{zz} \underline{e}_z + \tau_{\theta z} \underline{e}_\theta \right] \right) \cdot \underline{e}_z \, dA$$

$$= \iint_{A_o} r \tau_{\theta z} \, dA$$

$$= \iint_{A_o} r^2 \frac{G\Omega}{L} \, dA = \frac{G\Omega}{L} \underbrace{\iint_{A_o} r^2 \, dA}_{\text{Polar moment of inertia, } J}$$

The quantity $\iint_{A_o} r^2 \, dA$ is called polar moment of inertia (a geometric quantity)

So, finally torque T is equal to

$$\boxed{T = GJ \frac{\Omega}{L} \quad \text{or,} \quad \Omega = \frac{TL}{GJ}}$$

torque

torsional stiffness

twist

Thin cylinder approximation

For thin cylinder, $r_o \approx r_i = r_m$ (Outer wall radius $\approx$ Inner wall radius)

and

$$t = r_o - r_i$$

The radial and hoop stress then become:

$$\sigma_{rr}\,(r = r_i) = -P_i$$

$$\sigma_{\theta\theta}\,(r = r_i) = \frac{P_i\, r_i^2}{r_o^2 - r_i^2}\left(1 + \frac{r_o^2}{r_i^2}\right) = \frac{P_i}{\underbrace{(r_o - r_i)}_{t}}\ \frac{\overbrace{(r_o^2 + r_i^2)}^{2r_m^2}}{\underbrace{(r_o + r_i)}_{2r_m}}$$

$$= \frac{P_i\, r_m}{t}$$

For $t \ll r_m \Rightarrow \dfrac{r_m}{t} \gg 1$

So, $\sigma_{\theta\theta} \gg \sigma_{rr}$ $\rightarrow$ Hence, in comparison to $\sigma_{\theta\theta}$, the small radial stress component is normally ignored