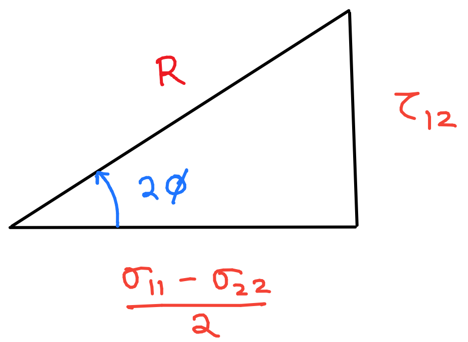
From the 2D plane stress case, we found that the normal and shear stress components acting on an inclined plane with normal $\underline{n}$ (inclined at an angle θ with $\underline{e}_1$) are given by

$$\sigma_n = \frac{\sigma_{11} + \sigma_{22}}{2} + \frac{\sigma_{11} - \sigma_{22}}{2} \cos 2\theta + \tau_{12} \sin 2\theta$$

$$\tau_n = -\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right) \sin 2\theta + \tau_{12} \cos 2\theta$$

Now, let us define a scalar $R = \sqrt{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \tau_{12}^2}$

We will think of this R as the magnitude of the hypotenuse of a right-angled triangle with base as $\frac{\sigma_{11} - \sigma_{22}}{2}$ and height as τ_{12}



From trigonometry, you can see that

$$\sin 2\phi = \frac{\tau_{12}}{R}$$

$$\cos 2\phi = \frac{\sigma_{11} - \sigma_{22}}{2R}$$

Using R, we can rewrite σ_n and τ_n as:

$$\sigma_n = \frac{\sigma_{11} + \sigma_{22}}{2} + R\left(\cos 2\phi \cos 2\theta + \sin 2\phi \sin 2\theta\right)$$

$$\Rightarrow \sigma_n = \frac{\sigma_{11} + \sigma_{22}}{2} + R \cos(2\phi - 2\theta)$$

$$\tau_n = R\left(-\cos 2\phi \sin 2\theta + \sin 2\phi \cos 2\theta\right)$$

$$\Rightarrow \tau_n = R \sin(2\phi - 2\theta)$$

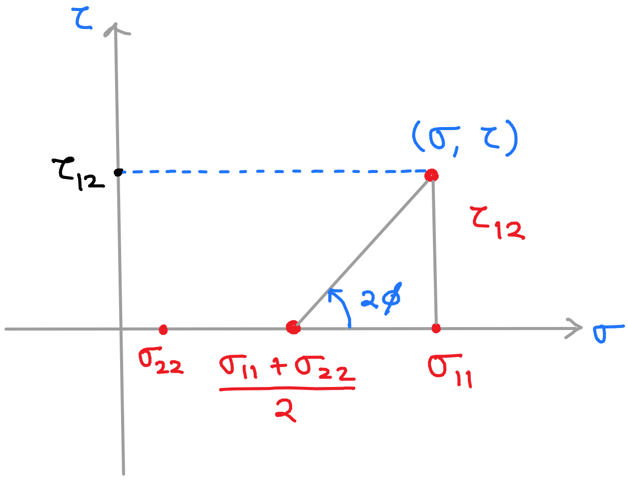
Mohr's circle

$$\sigma_n = \frac{\sigma_{11} + \sigma_{22}}{2} + R\cos(2\phi - 2\theta)$$

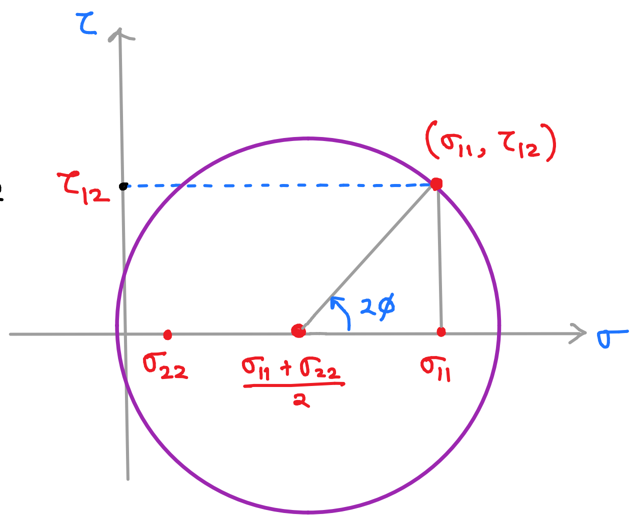
$$\tau_n = R\sin(2\phi - 2\theta)$$

Based on these formulas, lets try to obtain the locus of σ_n and τ_n for all values of θ.

Let us think of a $\sigma - \tau$ plane and plot σ_n and τ_n for each value of θ in this plane. The plane has σ on x-axis and τ on the y-axis.

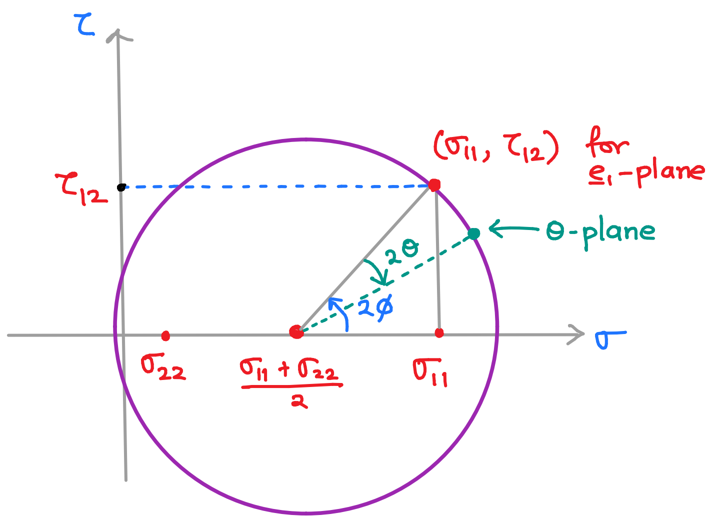


From the above relations, one can figure out that the centre of the circle on the σ-axis will be at $\left(\frac{\sigma_{11} + \sigma_{22}}{2}, 0\right)$. We can place σ_{11} and σ_{22} on the σ-axis and τ_{12} on the τ-axis. Then, we plot the point (σ_{11}, τ_{12}) which corresponds to $\underline{e}_1$-plane. If you join this point wi the center, the line obtained will give us the radius of the circle, which turns out to be R.



Once we obtain the radius and center of the circle, we can draw the complete circle $\longrightarrow$ is called the Mohr's circle

The 2D Mohr's circle can be used to find the 2D state of stress on any plane.



$$\sigma_n = \frac{\sigma_{11}+\sigma_{22}}{2} + R\cos(2\phi - 2\theta)$$
$$\tau_n = R\sin(2\phi - 2\theta)$$

While we can use the above equations for finding σ and τ on any arbitrary plane at an angle θ. For using the Mohr's circle, we note the angle in the cosine and sine terms is $2\phi - 2\theta$

So the radial line from the center to the point corresponding to the θ-plane on the Mohr's circle should be at an angle of $(2\phi - 2\theta)$ from the $\underline{e}_1$-plane. In other words, we can obtain the point corresponding to the θ-plane on Mohr's circle by going in the clockwise direction by angle 2θ from the $\underline{e}_1$-plane point. So the radial line from the center to the point

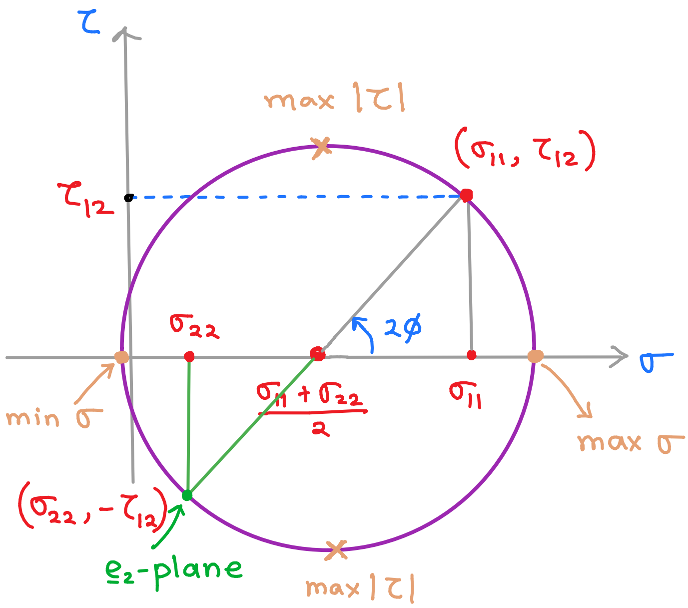
Steps for drawing Mohr's circle and for finding the point corresponding to θ-plane

1. Draw the center of the circle at $\left(\frac{\sigma_{11}+\sigma_{22}}{2}, 0\right)$
2. Draw (σ, τ) for $\underline{e}_1$-plane, i.e., the point (σ_{11}, τ_{12})
3. Draw a line joining the center and the point (σ_{11}, τ_{12}) to get the radius of the circle
4. With the center and radius known, draw the Mohr's circle
5. To find (σ, τ) for θ-plane, rotate the radial line of $\underline{e}_1$-plane by 2θ clockwise

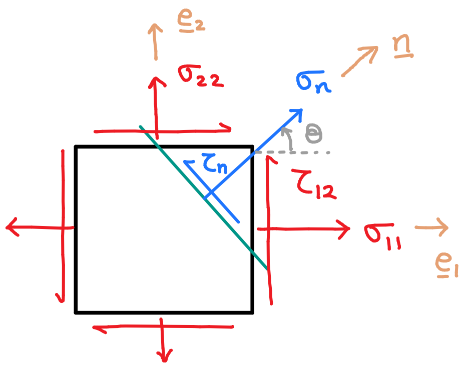
Notice that the normal to the θ-plane is at angle θ with $\underline{e}_1$ in counterclockwise direction. But on the Mohr's circle, we draw that point by rotating 2θ in clockwise direction from the point corresponding to $\underline{e}_1$-plane. This is because we have 2θ with a minus sign in the argument of trigonometric functions

Sign Convention while using Mohr's circle

Once we have determined the point on Mohr's circle corresponding to $\underline{e}_1$-plane, we can obtain the $(\sigma-\tau)$ corresponding to the $\underline{e}_2$-plane by rotating $2\times 90°$ in the clockwise direction from $\underline{e}_1$-plane. Thus we get the $\underline{e}_2$-plane at the diametrically opposite point w.r.t. the $\underline{e}_1$-plane.



The point for $\underline{e}_2$-plane has coordinates $(\sigma_{22}, -\tau_{12})$. However, we know the shear stress on the $\underline{e}_2$-plane is τ_{12}. So why are we getting $-\tau_{12}$ from the Mohr's circle? This is because of our convention for the sign of τ_n is taken as +ve if τ_n is acting $90°$ ccw direction from its plane normal $\underline{n}$.

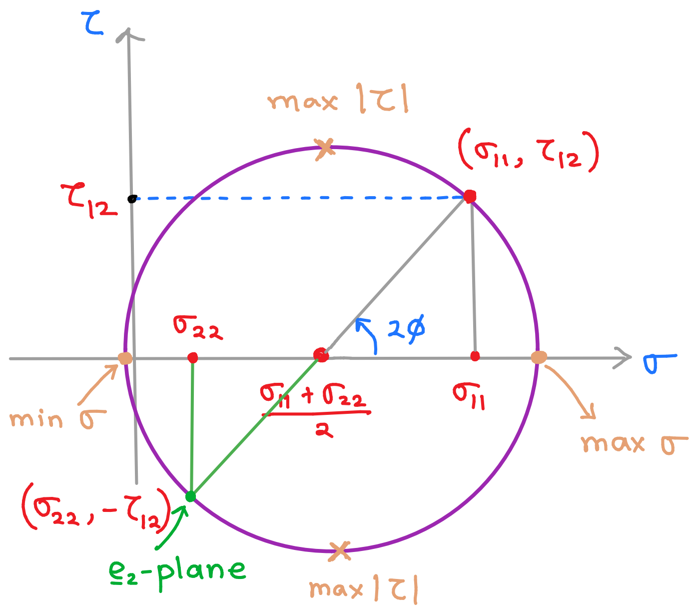


For $\underline{e}_2$-plane, if we go $90°$ in the CCW direction from its plane normal $\underline{e}_2$, we would be pointing towards $-\underline{e}_1$. So, Mohr's circle is giving us τ on $\underline{e}_2$-plane along $-\underline{e}_1$ direction, whereas τ_{12}, by definition, is the shear component along $+\underline{e}_1$-direction. Therefore, Mohr's circle gives us $-\tau_{12}$ for shear traction on $\underline{e}_2$-plane.

Other conclusions that can be drawn using Mohr's circle

We can get the maximum and minimum values of σ and τ

- The maximum/minimum values of σ are plotted on σ-axis itself. The maximum value of σ will correspond to the principal stress λ_1 and the min value of σ to λ_2



$$\lambda_1 = \frac{\sigma_{11} + \sigma_{22}}{2} + R \, , \quad \lambda_2 = \frac{\sigma_{11} + \sigma_{22}}{2} - R$$

This allows us to get the values of principal stress components directly from the Mohr's circle. One can also write the center and radius of the circle in terms of principal stresses λ_1 and λ_2

$$\text{Center} = \left(\frac{\lambda_1 + \lambda_2}{2}, \, 0 \right), \quad \text{Radius} = \frac{\lambda_1 - \lambda_2}{2}$$

- The max/min values of shear are equal to the radius of the Mohr's circle