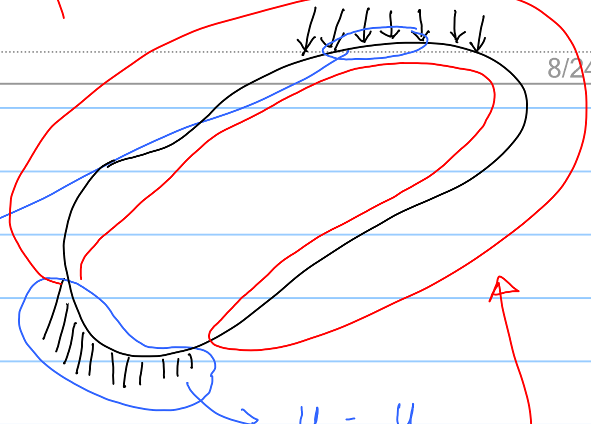
Lecture 9

$\nabla \cdot \underline{\underline{\sigma}}$ — 3+3 equation!

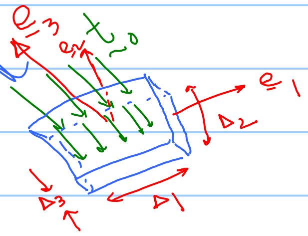
$$\boxed{\begin{aligned} \text{LMB:} &\quad \sum \frac{\partial \underline{\underline{\sigma}}}{\partial x_i}\underline{e}_i + \underline{b} = \rho \underline{a} \\ \text{AMB:} &\quad \underline{\underline{\sigma}} = \underline{\underline{\sigma}}^T \end{aligned}}$$

Governing equation + B.C.s.



→ displacement boundary condition — $\underline{u} = \underline{u}_0$

→ traction boundary condition



external load — Traction forces

Force balance :- $\underline{t}^0 \Delta_1\Delta_2 + \underline{t}^{-3}\Delta_1\Delta_2 + \underline{t}^1 \Delta_2\Delta_3 + \underline{t}^{-1}\Delta_2\Delta_3 + (\underline{t}^2 + \underline{t}^{-2})\Delta_1\Delta_3$
$+ \underline{b}\,\Delta_1\Delta_2\Delta_3 = \rho\,\Delta_1\Delta_2\Delta_3\,\underline{a}$

Dividing by $\Delta_1\Delta_2$

$$\underline{t}^0 + \underline{t}^{-3} + (\underline{t}^1 + \underline{t}^{-1})\cancel{\frac{\Delta_3}{\Delta_1}} + (\underline{t}^2 + \underline{t}^{-2})\cancel{\frac{\Delta_3}{\Delta_2}} + \cancel{(\underline{b} - \rho\underline{a})\Delta_3} = \underline{0}$$

$\lim_{\Delta_3 \to 0}$ — bringing the bottom surface infinitesimally close to the exposed surface

$$\Rightarrow \underline{t}^{-3} = -\underline{t}^0$$

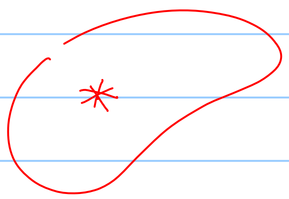
$$\Rightarrow \boxed{\underline{t}^3 = \underline{t}^0}$$

$$\Rightarrow \boxed{\underline{\underline{\sigma}}\,\underline{n} = \underline{t}^0}$$ → traction boundary condition!

$\underline{n}$: outward normal to surface

$$[\underline{\underline{\sigma}}]_{(\underline{e}_1, \underline{e}_2, \underline{e}_3)} = \begin{bmatrix} \times & \times & t^0_1 \\ \times & \times & t^0_2 \\ t^0_1 & t^0_2 & t^0_3 \end{bmatrix}$$

Principal planes / principal stress component



→ Max/min of normal components of tractions at a pt!

$$\underline{\underline{\sigma}}, \quad \underline{t}^n = \underline{\underline{\sigma}}\,\underline{n}, \quad \sigma_{nn} = \underline{t}^n \cdot \underline{n} = (\underline{\underline{\sigma}}\,\underline{n}) \cdot \underline{n}$$

$$\sigma_{nn} = \left(\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \right) \cdot \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

$$= \sum_i \sum_j \sigma_{ij} n_j n_i$$

(n_1, n_2, n_3 are unknowns)

$n_1^2 + n_2^2 + n_3^2 = 1 \Rightarrow n_3 = \sqrt{1 - n_1^2 - n_2^2}$

$$f(n_1, n_2, n_3, \lambda) = \sigma_{nn} - \lambda \left(n_1^2 + n_2^2 + n_3^2 - 1 \right)$$

$$= \sum_i \sum_j \sigma_{ij} n_i n_j - \lambda \left(\sum_i n_i n_i - 1 \right)$$

$$\frac{\partial f}{\partial n_k} = \sum_i \sum_j \sigma_{ij} \left(\underbrace{\frac{\partial n_i}{\partial n_k}}_{\delta_{ik}} n_j + n_i \underbrace{\frac{\partial n_j}{\partial n_k}}_{\delta_{jk}} \right) - \lambda \sum_i 2 n_i \underbrace{\frac{\partial n_i}{\partial n_k}}_{\delta_{ik}}$$

$$= \sum_j \sigma_{kj} n_j + \underbrace{\sum_i \sigma_{ik} n_i}_{} - 2\lambda n_k$$

$$= \sum_j \sigma_{kj} n_j + \sum_j \sigma_{jk} n_j - 2\lambda n_k$$

$$= \sum_j (\sigma_{kj} + \sigma_{jk}) n_j - 2\lambda n_k$$

$$= 2 \sum_j \sigma_{kj} n_j - 2\lambda n_k$$

$$\therefore \frac{\partial f}{\partial n_k} = 2 \left(\sum_j \sigma_{kj} n_j - \lambda n_k \right) = 0$$

$$\frac{\partial f}{\partial \lambda} = \sum n_i n_i - 1$$

$$[\underline{\sigma}][\underline{n}] = \lambda [\underline{n}]$$

Eigenvalue - Eigenvector equation!

eigenvector

eigenvalue

Unit-normed eigenvectors: principal planes!

eigenvalues: principal stress components!

$$[\underline{t}^n] = [\underline{\sigma}][\underline{n}] = \lambda [\underline{n}]$$

$\Rightarrow$ traction on principal planes have no shear component)