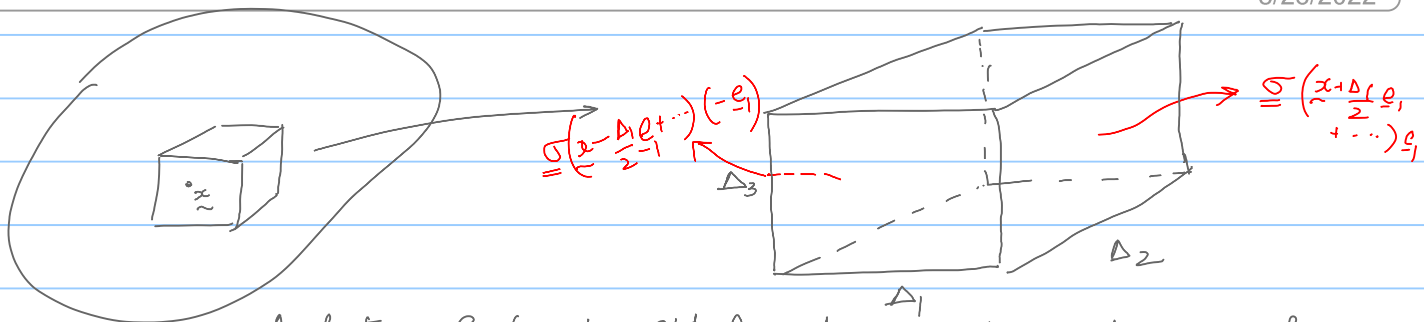
Lecture 8 Stress equilibrium equations.

Note Title 8/23/2022



Application of Newton's 2^{nd} law to cuboid + shrinking of cuboid to $\underset{\sim}{x}$ $(\Delta V \to 0)$

$$\Rightarrow \boxed{\sum_{i=1}^{3} \frac{\partial \underset{\approx}{\sigma}}{\partial x_i} \underset{\sim}{e}_i + \underset{\sim}{b} = \rho \underset{\sim}{a}}$$

Linear momentum balance (LMB)

$\underset{\sim}{\nabla} \cdot \underset{\approx}{\sigma}$

Write it in the coordinate system of $(\underset{\sim}{e}_1 - \underset{\sim}{e}_2 - \underset{\sim}{e}_3)$

$$\left[\frac{\partial \underset{\approx}{\sigma}}{\partial x_1} \underset{\sim}{e}_1\right]_{(e_1 - e_2 - e_3)} \rightarrow \begin{bmatrix} \frac{\partial \sigma_{11}}{\partial x_1} \\ \frac{\partial \tau_{21}}{\partial x_1} \\ \frac{\partial \tau_{31}}{\partial x_1} \end{bmatrix} + \begin{bmatrix} \frac{\partial \tau_{12}}{\partial x_2} \\ \frac{\partial \sigma_{22}}{\partial x_2} \\ \frac{\partial \tau_{32}}{\partial x_2} \end{bmatrix} + \begin{bmatrix} \frac{\partial \tau_{13}}{\partial x_3} \\ \frac{\partial \tau_{23}}{\partial x_3} \\ \frac{\partial \sigma_{33}}{\partial x_3} \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} \rho a_1 \\ \rho a_2 \\ \rho a_3 \end{bmatrix}$$

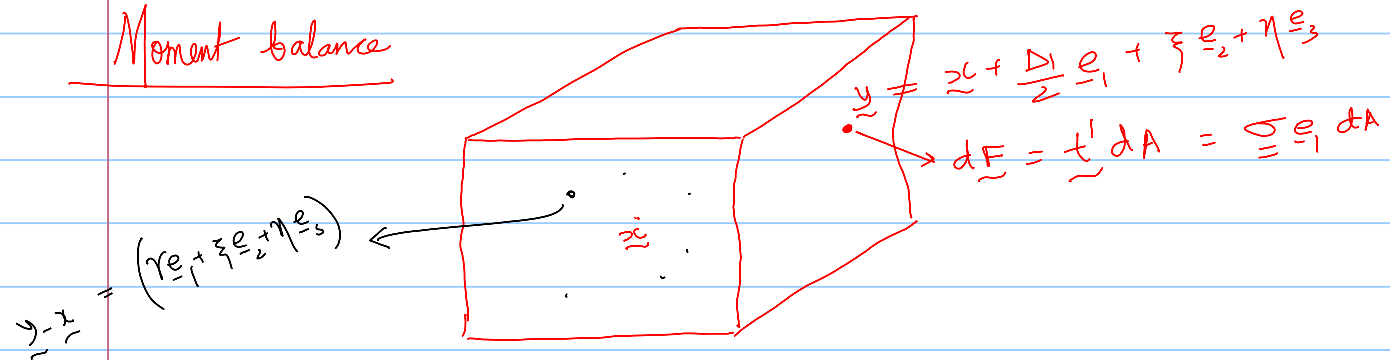
$$\boxed{\begin{aligned} \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \tau_{12}}{\partial x_2} + \frac{\partial \tau_{13}}{\partial x_3} + b_1 &= \rho a_1 \\ \frac{\partial \tau_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \tau_{23}}{\partial x_3} + b_2 &= \rho a_2 \\ \frac{\partial \tau_{31}}{\partial x_1} + \frac{\partial \tau_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + b_3 &= \rho a_3 \end{aligned}}$$

→ Force balance along e_1

→ Force balance along e_2

→ along e_3

Moment balance



$$\sum \underset{\sim}{M}^{ext}_{/\underset{\sim}{x}} = \frac{d}{dt}\left(\underset{\sim}{H}_{/\underset{\sim}{x}}\right)\Big|_{C.M}$$

center of mass

$$\underset{\sim}{M}^{+1} = \iint (\underset{\sim}{y} - \underset{\sim}{x}) \times \underset{\approx}{\sigma}(\underset{\sim}{y})\, \underset{\sim}{e}_1 \, dA$$

$$= \iint \left(\frac{\Delta_1}{2}\underset{\sim}{e}_1 + \xi\, \underset{\sim}{e}_2 + \eta\, \underset{\sim}{e}_3\right) \times \left[\underset{\approx}{\sigma}(\underset{\sim}{x}) + \frac{\partial \underset{\approx}{\sigma}}{\partial x_1}\frac{\Delta_1}{2} + \frac{\partial \underset{\approx}{\sigma}}{\partial x_2}\xi + \frac{\partial \underset{\approx}{\sigma}}{\partial x_3}\eta\right]\underset{\sim}{e}_1 \, d\xi\, d\eta$$

$$= \underbrace{\frac{\Delta_1}{2}\, \underset{\sim}{e}_1 \times \underset{\approx}{\sigma}\, \underset{\sim}{e}_1\, \Delta_2 \Delta_3}_{\Delta V} + \underbrace{\frac{\Delta_1^2}{4}\, \underset{\sim}{e}_1 \times \frac{\partial \underset{\approx}{\sigma}}{\partial x_1} \underset{\sim}{e}_1\, \Delta_2 \Delta_3}_{\Delta V \Delta_1 \;\to\; o(\Delta V)} + \cdots$$

$$= \underset{\sim}{e}_1 \times \underset{\approx}{\sigma}\, \underset{\sim}{e}_1 \frac{\Delta V}{2} + o(\Delta V)$$

$$\underset{\sim}{M}^{-1} = \underset{\sim}{e}_1 \times \underset{\approx}{\sigma}\, \underset{\sim}{e}_1 \frac{\Delta V}{2} + o(\Delta V)$$

$$\underset{\sim}{M}^{1} + \underset{\sim}{M}^{-1} = \underset{\sim}{e}_1 \times \underset{\approx}{\sigma}\, \underset{\sim}{e}_1\, \Delta V + o(\Delta V)$$

$$\Rightarrow \sum_{i=1}^{3} \underset{\sim}{M}^{i} + \underset{\sim}{M}^{-i} = \left(\sum_{i=1}^{3} \underset{\sim}{e}_i \times \underset{\approx}{\sigma}\, \underset{\sim}{e}_i\right) \Delta V + o(\Delta V)$$

$$\underset{\sim}{M}^{b}_{/\underset{\sim}{x}} = \iiint (\underset{\sim}{y} - \underset{\sim}{x}) \times \underset{\sim}{b}(\underset{\sim}{y})\, dV = \iiint (r\, \underset{\sim}{e}_1 + \xi\, \underset{\sim}{e}_2 + \eta\, \underset{\sim}{e}_3) \times \underset{\sim}{b}(\underset{\sim}{x})\, dV$$

$$\underset{\sim}{b}(\underset{\sim}{x}) \to \left[\underset{\sim}{b}(\underset{\sim}{x}) + \frac{\partial \underset{\sim}{b}}{\partial x_1} r + \frac{\partial \underset{\sim}{b}}{\partial x_2}\xi + \frac{\partial \underset{\sim}{b}}{\partial x_3}\eta + \cdots\right]$$

$$= o(\Delta V)$$

$o(\Delta V)$ $\quad 0 \quad 0 \quad 0$

velocity w.r.t. center of mass

$$\underset{\sim}{H}_{/\underset{\sim}{x}} = \iiint_{\Delta V} (\underset{\sim}{y} - \underset{\sim}{x}) \times \rho\, \underset{\sim}{v}_{/\underset{\sim}{x}}\, dV$$

$$\frac{d}{dt}\left(\underset{\sim}{H}_{/\underset{\sim}{x}}\right) = \frac{d}{dt}\iiint_{\Delta V} \cdots\cdots = \frac{d}{dt}\iiint_{\Delta m} (\underset{\sim}{y} - \underset{\sim}{x}) \times \underset{\sim}{v}_{/\underset{\sim}{x}}\, dm$$

Control vol. ↳ a function of time

Control mass

$$= \iiint_{\Delta m} \left[\frac{d}{dt}(\underset{\sim}{y} - \underset{\sim}{x}) \times \underset{\sim}{v}_{/\underset{\sim}{x}} + (\underset{\sim}{y} - \underset{\sim}{x}) \times \frac{d}{dt}\left(\underset{\sim}{v}_{/\underset{\sim}{x}}\right)\right] dm$$

$$= \iiint_{\Delta m} \left[\cancel{\underline{v}_{/x} \times \underline{v}_{/x}} + (\underline{y} - \underline{x}) \times \underline{a}_{/x} \right] dm$$

$$= \iiint_{\Delta m} (\underline{y} - \underline{x}) \times \underline{a}_{/x} \, dm$$

$$= \iiint_{\Delta V} (\underline{y} - \underline{x}) \times \underline{a}_{/x} \, \rho \, dV \quad \to b$$

$$= o(\Delta V)$$

$$\Rightarrow \lim_{\Delta V \to 0} \frac{\sum_{i=1}^{3} \underline{e}_i \times \underline{\underline{\sigma}}\, \underline{e}_i \, \Delta V + o(\Delta V) + o(\Delta V) = o(\Delta V)}{\Delta V}$$

$$\Rightarrow \boxed{\sum_{i=1}^{3} \underline{e}_i \times \underline{\underline{\sigma}}\, \underline{e}_i = \underline{0}} \quad \to \text{Angular Momentum balance AMB!}$$

$(\underline{e}_1 - \underline{e}_2 - \underline{e}_3)$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} \sigma_{11} \\ \tau_{21} \\ \tau_{31} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} \tau_{12} \\ \sigma_{22} \\ \tau_{32} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} \tau_{13} \\ \tau_{23} \\ \sigma_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 \\ -\tau_{31} \\ \tau_{21} \end{bmatrix} + \cdots$$

$$= \begin{bmatrix} \tau_{23} - \tau_{32} \\ \tau_{13} - \tau_{31} \\ \tau_{21} - \tau_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Downarrow$

Stress matrix is symmetric

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Stress tensor is sy...