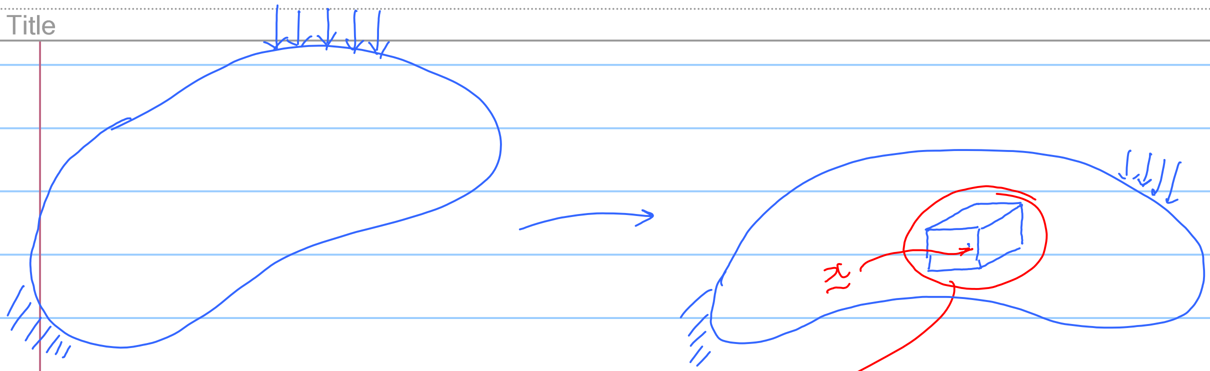
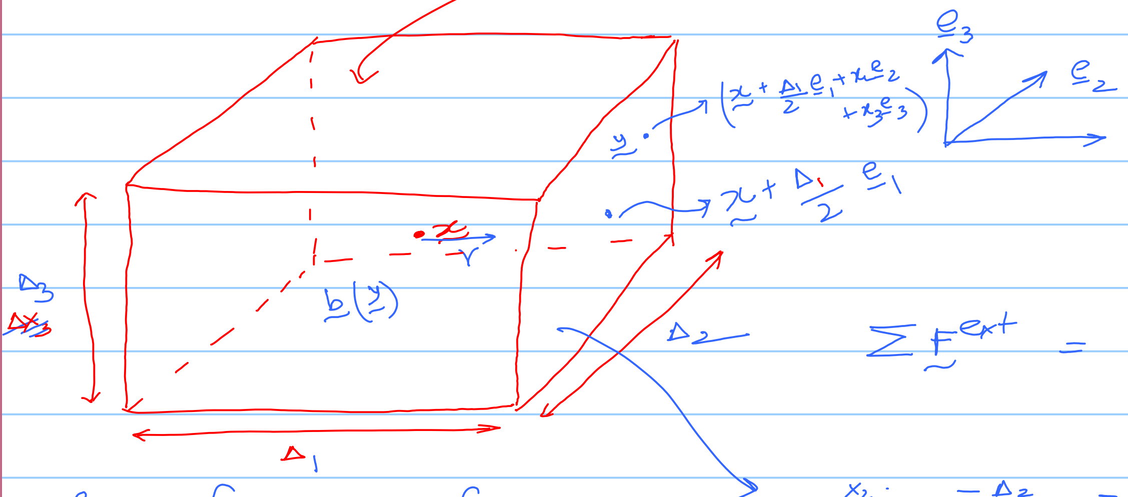
Lecture 7 Stress equilibrium equations

Note Title 8/17/2022



→ how to obtain the final configuration of the body?

→ how to obtain the distribution of traction / stress tensor in the body?



$$\sum F^{ext} = \frac{d}{dt}(P)$$

traction force on e_1- face.

$$F^1 = \int_{-\Delta_2/2}^{\Delta_2/2}\int_{-\Delta_3/2}^{\Delta_3/2} t^1(y)\, dx_2\, dx_3$$

$x_2 : \ -\frac{\Delta_2}{2} \to \frac{\Delta_2}{2}$

$x_3 : \ -\frac{\Delta_3}{2} \to \frac{\Delta_3}{2}$

$$= \iint \sigma(y)\, e_1\, dx_2\, dx_3$$

$$= \iint \left[\sigma(x) + \frac{\partial \sigma}{\partial x_1}(x)\frac{\Delta_1}{2} + \frac{\partial \sigma}{\partial x_2}(x)\, x_2 + \frac{\partial \sigma}{\partial x_3}(x)\, x_3 + \dots \right] e_1\, dx_2\, dx_3$$

$$= \sigma(x)\, e_1\, \Delta_2 \Delta_3 + \frac{\partial \sigma}{\partial x_1}\frac{\Delta_1}{2}\, e_1\, \Delta_2\Delta_3 + \frac{\partial \sigma}{\partial x_2}(x)\, e_1 \cancel{\iint x_2\, dx_2\, dx_3} \to 0$$

$$\frac{x_2^2}{2}\Big|_{-\Delta_2/2}^{\Delta_2/2} \int dx_3 \to 0$$

$$= \sigma(x)\, e_1\, \Delta_2\Delta_3 + \frac{\partial \sigma}{\partial x_1}\, e_1\, \frac{\Delta_1\Delta_2\Delta_3}{2} + \dots$$

$$F^{-1} = \iint t^{-1}\, dx_2\, dx_3$$

$$= \iint t\left(x - \frac{\Delta_1}{2} e_1 + x_2 e_2 + x_3 e_3\right) dx_2\, dx_3$$

$$= \iint \sigma\left(x - \frac{\Delta_1}{2} e_1 + x_2 e_2 + x_3 e_3\right)(-e_1)\, dx_2\, dx_3$$

$$= -\underline{\underline{\sigma}}(\underline{x})\,\underline{e}_1\,\Delta_2\Delta_3 + \frac{\partial \underline{\underline{\sigma}}}{\partial x_1}\left(-\frac{\Delta_1}{2}\right)(-\underline{e}_1)\,\Delta_2\Delta_3 + \ldots$$

$$= -\underline{\underline{\sigma}}(\underline{x})\,\underline{e}_1\,\Delta_2\Delta_3 + \frac{\partial \underline{\underline{\sigma}}}{\partial x_1}\frac{\underline{e}_1\Delta_1\Delta_2\Delta_3}{2}$$

$$\underline{F}^1 + \underline{F}^{-1} = \frac{\partial \underline{\underline{\sigma}}}{\partial x_1}(\underline{x})\,\underline{e}_1\,\Delta_1\Delta_2\Delta_3 = \frac{\partial \underline{\underline{\sigma}}}{\partial x_1}(\underline{x})\,\underline{e}_1\,\Delta V + o(\Delta V)$$

$$\lim_{\Delta V\to 0}\frac{o(\Delta V)}{\Delta V} = 0$$

$$\Rightarrow \sum_{i=1}^{3} \underline{F}^i + \underline{F}^{-i} = \sum_{i=1}^{3}\frac{\partial \underline{\underline{\sigma}}}{\partial x_i}\,\underline{e}_i\,\Delta V + o(\Delta V)$$

Body force

$$\underline{y} = \underline{x} + x_1\underline{e}_1 + x_2\underline{e}_2 + x_3\underline{e}_3$$

$$\iiint \underline{b}(\underline{y})\,dx_1dx_2dx_3 = \iiint\left[\underline{b}(\underline{x}) + \frac{\partial \underline{b}}{\partial x_1}x_1 + \frac{\partial \underline{b}}{\partial x_2}x_2 + \frac{\partial \underline{b}}{\partial x_3}x_3 + \ldots\right]dV$$

$$= \underline{b}(\underline{x})\,\Delta V + o(\Delta V)$$

$$\underline{P} = \iiint_V \rho(\underline{y})\,\underline{v}(\underline{y})\,dV = \iiint_M dm\,\underline{a}(\underline{y})$$

$$\frac{d}{dt}\underline{P} = \frac{d}{dt}\left[\iiint_{V(t)} \rho(\underline{y})\,\underline{v}(\underline{y})\,dV\right]$$

$$= \frac{d}{dt}\left[\iiint_M dm\,\underline{v}\right] = \iiint_M dm\,\underline{a} = \iiint_V \rho(\underline{y})\,\underline{a}(\underline{y})\,dV$$

$$= \rho(\underline{x})\,\underline{a}(\underline{x})\,\Delta V + o(\Delta V)$$

$$\lim_{\Delta V\to 0}\frac{\sum_{i=1}^{3}\frac{\partial \underline{\underline{\sigma}}}{\partial x_i}(\underline{x})\,\underline{e}_i\,\Delta V + o(\Delta V) + \underline{b}(\underline{x})\,\Delta V + o(\Delta V) = \rho(\underline{x})\,\underline{a}(\underline{x})\,\Delta V + o(\Delta V)}{\Delta V}$$

$$\boxed{\sum_{i=1}^{3}\frac{\partial \underline{\underline{\sigma}}}{\partial x_i}\,\underline{e}_i + \underline{b}(\underline{x}) = \rho(\underline{x})\,\underline{a}(\underline{x})}$$