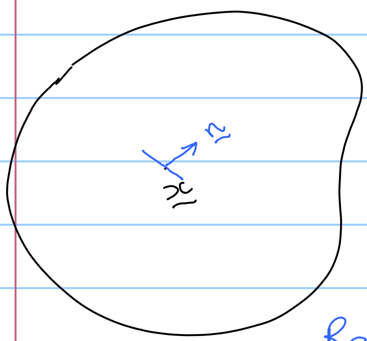
Lecture 6 Stress Tensor

$$\underset{\sim}{t}^{n}(\underset{\sim}{x}) = \sum_{i=1}^{3} \underset{\sim}{t}^{i} (\underset{\sim}{n}\cdot\underset{\sim}{e}_i) \Rightarrow \underset{\sim}{t}^{n}(\underset{\sim}{x}) = \left(\sum_{i=1}^{3} \underset{\sim}{t}^{i}\otimes\underset{\sim}{e}_i\right)\underset{\sim}{n}$$

$$= \sum_{i=1}^{3} \underset{\sim}{t}^{\hat{i}} (\underset{\sim}{n}\cdot\hat{\underset{\sim}{e}}_i) \qquad = \underset{\approx}{\sigma}(\underset{\sim}{x})\cdot\underset{\sim}{n}$$

$$= \left(\sum_{i=1}^{3} \underset{\sim}{t}^{\hat{i}}\otimes\hat{\underset{\sim}{e}}_i\right)\underset{\sim}{n} = \underset{\approx}{\sigma}\cdot\underset{\sim}{n}$$



Representation of stress tensor : Stress matrix in $(e_1 - e_2 - e_3)$ coord sys.

$$\sigma_{ij} = (\underset{\approx}{\sigma}\cdot\underset{\sim}{e}_j)\cdot\underset{\sim}{e}_i = \underset{\sim}{t}^{j}\cdot\underset{\sim}{e}_i$$

$$\begin{bmatrix}\sigma_{11} & \sigma_{12} & \sigma_{13}\\ \sigma_{21} & \sigma_{22} & \sigma_{23}\\ \sigma_{31} & \sigma_{32} & \sigma_{33}\end{bmatrix} = \begin{bmatrix} t^1\cdot e_1 & t^2\cdot e_1 & t^3\cdot e_1\\ t^1\cdot e_2 & t^2\cdot e_2 & t^3\cdot e_2\\ t^1\cdot e_3 & t^2\cdot e_3 & t^3\cdot e_3\end{bmatrix}$$

$[\underset{\sim}{t}^1]_{(e_1-e_2-e_3)}$ $\quad [\underset{\sim}{t}^2]_{(e_1-e_2-e_3)}$ $\quad [\underset{\sim}{t}^3]_{(e_1-e_2-e_3)}$

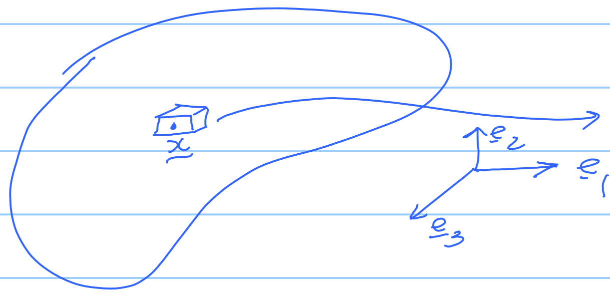
normal component of traction on e_1- plane

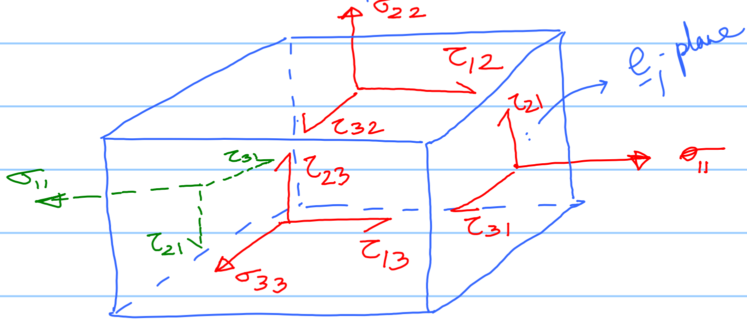
$$\| \qquad \begin{bmatrix}\sigma_{11} & \tau_{12} & \tau_{13}\\ \tau_{21} & \sigma_{22} & \tau_{23}\\ \tau_{31} & \tau_{32} & \sigma_{33}\end{bmatrix}$$

shear comp. of traction on e_1- plane along e_2

shear component of traction on e_1- plane along e_3

τ_{ij}: i → component direction! ; j → signifies plane normal





imaginary cube of side "0" at $\underset{\sim}{x}$!

$$\underset{\sim}{t}^{-1} = \underset{\approx}{\sigma}(-\underset{\sim}{e}_1) = -\underset{\approx}{\sigma}\,\underset{\sim}{e}_1$$

$$\Rightarrow [\underset{\sim}{t}^{-1}] = -\begin{bmatrix}\sigma_{11}\\ \tau_{21}\\ \tau_{31}\end{bmatrix}$$

Transformation of stress matrix

$$\underset{\approx}{\sigma} = \sum_i \sum_j \sigma_{ij}\, \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j = \sum_i \sum_j \hat{\sigma}_{ij}\, \hat{\underset{\sim}{e}}_i \otimes \hat{\underset{\sim}{e}}_j$$

$$\hat{\sigma}_{ij} \overset{?}{=} \sigma_{ij}$$

$$\begin{aligned}
\hat{\sigma}_{ij} &= \left(\underset{\approx}{\sigma} \cdot \hat{\underset{\sim}{e}}_j\right) \cdot \hat{\underset{\sim}{e}}_i \\
&= \hat{\underset{\sim}{t}}_j \cdot \hat{\underset{\sim}{e}}_i \\
&= \left(\sum_{k=1}^{3} \underset{\sim}{t}^k (\hat{\underset{\sim}{e}}_j \cdot \underset{\sim}{e}_k)\right) \cdot \hat{\underset{\sim}{e}}_i \\
&= \left(\sum_k \left(\sum_l (\underset{\sim}{t}^k \cdot \underset{\sim}{e}_l)\, \underset{\sim}{e}_l\right) (\hat{\underset{\sim}{e}}_j \cdot \underset{\sim}{e}_k)\right) \cdot \hat{\underset{\sim}{e}}_i \\
&= \left(\sum_k \sum_l t^k_l\, \underset{\sim}{e}_l\, (\hat{\underset{\sim}{e}}_j \cdot \underset{\sim}{e}_k)\right) \cdot \hat{\underset{\sim}{e}}_i
\end{aligned}$$

$$\begin{aligned}
\hat{\sigma}_{ij} &= \sum_k \sum_l \sigma_{lk}\, (\hat{\underset{\sim}{e}}_j \cdot \underset{\sim}{e}_k)\, (\hat{\underset{\sim}{e}}_i \cdot \underset{\sim}{e}_l) \\
&= \sum_k \sum_l \sigma_{lk}\, R_{kj}\, R_{li} \\
&= \sum_k \sum_l R^T_{il}\, \sigma_{lk}\, R_{kj}
\end{aligned}$$

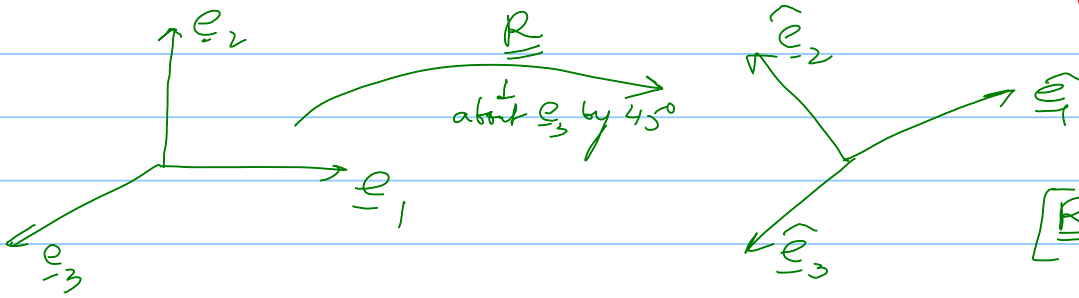
$$\boxed{\begin{aligned} \hat{\underset{\sim}{e}}_i &= \underset{\approx}{R}\, \underset{\sim}{e}_i \\ R_{ij} &= (\underset{\approx}{R} \cdot \underset{\sim}{e}_j) \cdot \underset{\sim}{e}_i = \hat{\underset{\sim}{e}}_j \cdot \underset{\sim}{e}_i \end{aligned}}$$

$$\Rightarrow \left[\hat{\sigma}_{ij}\right] = \left[R^T_{il}\right] \left[\sigma_{lk}\right] \left[R_{kj}\right]$$

$$[\underset{\approx}{D}] = [\underset{\approx}{A}][\underset{\approx}{B}][\underset{\approx}{C}] \qquad D_{ij} = \sum_{k}\sum_{l} A_{ik} B_{kl} C_{lj}$$

$$\Rightarrow \boxed{\left[\underset{\approx}{\sigma}\right]_{(\hat{e}_1-\hat{e}_2-\hat{e}_3)} = \underbrace{\left[\underset{\approx}{R}\right]^T \left[\underset{\approx}{\sigma}\right] \left[\underset{\approx}{R}\right]}_{(e_1-e_2-e_3)}}$$

$$\Rightarrow \left[\underset{\approx}{R}\right]_{(\hat{e}_1-\hat{e}_2-\hat{e}_3)} = \underbrace{\left[\underset{\approx}{R}\right]^T \left[\underset{\approx}{R}\right] \left[\underset{\approx}{R}\right]}_{(e_1-e_2-e_3)} = \left[\underset{\approx}{R}\right]_{(e_1 e_2 e_3)}$$



$$\left[\underset{\approx}{R}\right]_{(e_1-e_2-e_3)} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[\underline{\underline{\sigma}}]_{(\underline{e}_1-\underline{e}_2-\underline{e}_3)} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[\underline{\underline{\sigma}}]_{(\hat{\underline{e}}_1-\hat{\underline{e}}_2-\hat{\underline{e}}_3)} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 1/\sqrt{2} & 0 \\ -\frac{1}{\sqrt{2}} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[\underline{t}^{\hat{3}}]_{(\hat{\underline{e}}_1-\hat{\underline{e}}_2-\hat{\underline{e}}_3)} = [\underline{t}^{3}]_{(\hat{\underline{e}}_1-\hat{\underline{e}}_2-\hat{\underline{e}}_3)}$$