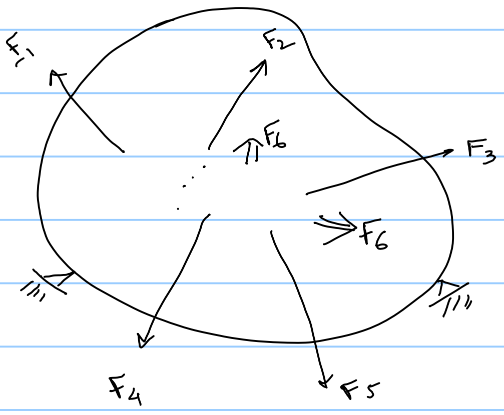
Lec 35 (Energy Methods)

Note Title 11/4/2022

Castigliano's first theorem :



$$E = \sum_i \frac{1}{2} F_i \delta_i$$

δ_i → corresponding displacement

$$= \sum_i \frac{1}{2} F_i \sum_j k_{ij} F_j$$

$$= \frac{1}{2} \sum_i \sum_j k_{ij} F_i F_j$$

$$\frac{\partial E}{\partial F_K} = \frac{1}{2} \sum_i \sum_j k_{ij} \left(\delta_{iK} F_j + F_i \delta_{jK} \right)$$

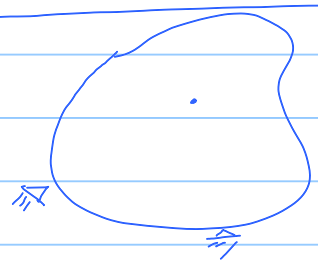
$$= \frac{1}{2} \sum_j k_{Kj} F_j + \frac{1}{2} \sum_i k_{iK} F_i$$

$$= \frac{1}{2} \underbrace{\sum_j k_{Kj} F_j}_{\delta_K} + \frac{1}{2} \underbrace{\sum_i k_{iK} F_i}_{\delta_K}$$

$k_{iK} \to k_{Ki}$

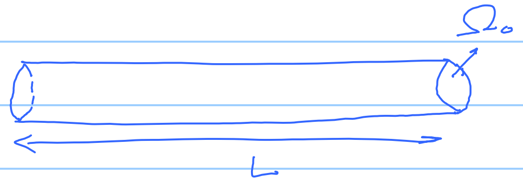
$$= \delta_K$$

$$\boxed{\delta_K = \frac{\partial E}{\partial F_K}}$$



$$\sum_i \sum_j \frac{1}{2} \sigma_{ij} \epsilon_{ij} \Rightarrow E = \sum_i \sum_j \iiint \frac{1}{2} \sigma_{ij} \epsilon_{ij} \, dv$$

$\sum\sum \frac{1}{2}\sigma_{ij}\epsilon_{ij}$ → Energy/vol.

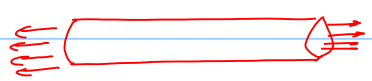


$$E = \sum_i \sum_j \int_0^L ds \iint_{\Omega_0} \frac{1}{2} \sigma_{ij} \epsilon_{ij} \, d\Omega_0$$

$\iint_{\Omega_0} \frac{1}{2} \sigma_{ij} \epsilon_{ij} \, d\Omega_0$ → cross-sectional energy (Energy/length)

$$E_{cs} = \sum_i \sum_j \iint_{\Omega_0} \frac{1}{2} \sigma_{ij} \epsilon_{ij} \, d\Omega_0$$

Pure extension.



$$[\sigma] = \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\sigma_{11} \to P/A$

$$[\epsilon] = \begin{bmatrix} \epsilon_{11} & 0 & 0 \\ 0 & \epsilon_{22} & 0 \\ 0 & 0 & \epsilon_{33} \end{bmatrix}$$

$\epsilon_{11} \to \sigma_{11}/E$

$$E_{cs} = \frac{1}{2} \iint_{\Omega_0} \sigma_{11} \epsilon_{11} \, d\Omega_0$$

$$= \frac{1}{2} \iint \frac{\sigma_{11}^2}{E} d\Omega_0 = \frac{1}{2} \frac{P^2}{A^2 E} \iint d\Omega_0 = \frac{P^2}{2AE}$$

AE → Extensional stiffness

Bending :

$$M_z = EI\kappa$$

$$\sigma_{xx} = -\frac{M_z y}{I}$$

$$\epsilon_{xx} = -y/R = -\kappa y = -\frac{M_z}{EI} y$$

$$E_{cs} = \frac{1}{2}\iint_{\Omega_0} \sigma_{xx}\epsilon_{xx}\, d\Omega_0 = \frac{1}{2}\iint_{\Omega_0} -\frac{M_z}{I} y \left(-\frac{M_z}{EI} y\right) d\Omega_0$$

$$= \frac{1}{2}\frac{M_z^2}{EI^2}\iint_{\Omega_0} y^2\, d\Omega_0$$

$$= \frac{1}{2}\frac{M_z^2}{EI_{zz}^2} I_{zz} = \frac{M_z^2}{2EI_{zz}}$$

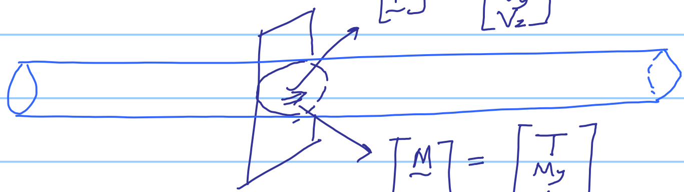
bending modulus

Energy due to torsion

$$E_{cs} = \frac{T^2}{2GJ}$$

Energy due to shear $= \dfrac{V^2}{2\kappa GA}$

$$[\underset{\sim}{F}] = \begin{bmatrix} P \\ V_y \\ V_z \end{bmatrix}$$



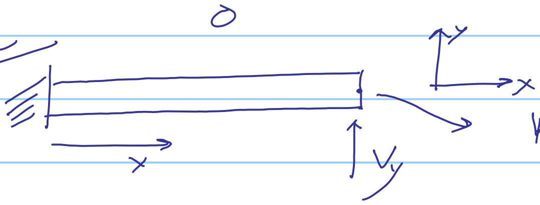
$$[\underset{\sim}{M}] = \begin{bmatrix} T \\ M_y \\ M_z \end{bmatrix}$$

If resolved along principal direction, total energy can be written to be simply the sum of individual energy (as if superposition holds)

$$E_{cs} = \frac{T^2(x)}{2GJ} + \frac{M_y^2(x)}{2EI_{yy}} + \frac{M_z^2(x)}{2EI_{zz}} + \frac{P^2(x)}{2EA} + \frac{V_y^2(x)}{2\kappa GA} + \frac{V_z^2(x)}{2\kappa GA}$$

$$E = \int_0^L E_{cs}\, dx$$

Example



How much is the tip rotation?

Q. How much is the tip deflection??

$$\delta = \frac{\partial E}{\partial V_y}$$

$P(x) = 0$ $\qquad M_y(x) = 0$

$V_y(x) = V_y$ $\qquad M_z(x) = V_y(L-x)$

$V_z(x) = 0$

$T(x) = 0$

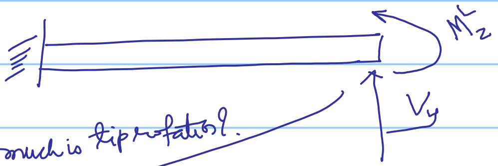
$$E = \int_0^L \left(\frac{V_y^2}{2\kappa GA} + \frac{V_y^2 (L-x)^2}{2EI_{zz}}\right) dx$$

$$\frac{\partial E}{\partial V_y} = \int_0^L \left(\frac{2V_y}{2\kappa GA} + \frac{2V_y (L-x)^2}{2EI_{zz}}\right) dx$$

$$\delta_y = \frac{V_y}{KGA} L + \frac{V_y}{E I_{zz}} \int_0^L (L-x)^2 dx$$

$$+ \frac{(x-L)^3}{3} \Big|_0^L = +L^3/3$$

$$\delta_y = \frac{V_y}{KGA} L + \frac{V_y L^3}{3 E I_{zz}}$$



How much is tip rotation?

$\theta_L \rightarrow$ Corresponding force is M_z^L

$$\theta_L = \frac{\partial E}{\partial M_z^L}$$

We are applying a dummy moment M_z^L

$P(x) = 0$

$V_y(x) = V_y$

$V_z(x) = 0$

$M_z(x) = M_z^L + V_y(L-x)$

$M_y(x) = 0$

$$E = \int_0^L \left[\frac{V_y^2}{2KGA} + \frac{\{M_z^L + V_y(L-x)\}^2}{2 E I_{zz}} \right] dx$$

$$\Rightarrow \frac{\partial E}{\partial M_z^L} = \int_0^L \frac{2\{M_z^L + V_y(L-x)\}}{2 E I_{zz}} dx$$

$$\Rightarrow \frac{\partial E}{\partial M_z^L} = \int_0^L \frac{2 V_y (L-x)}{2 E I_{zz}} dx$$

$$\theta_L = \frac{V_y}{E I_{zz}} \int_0^L (L-x)\, dx$$

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