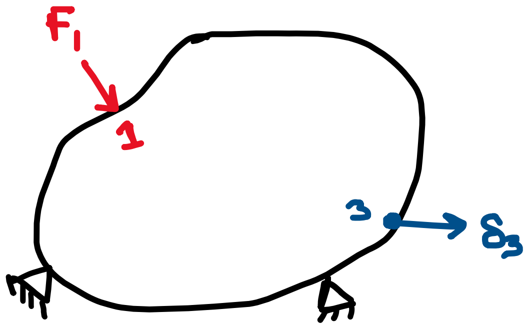
Classroom
Lecture 34

Introduction to Energy methods (contd…)

Concepts needed for energy methods

Recap of last lecture



- What is the relationship between the displacement δ_3 at point 3 and the force F_1 at point 1?

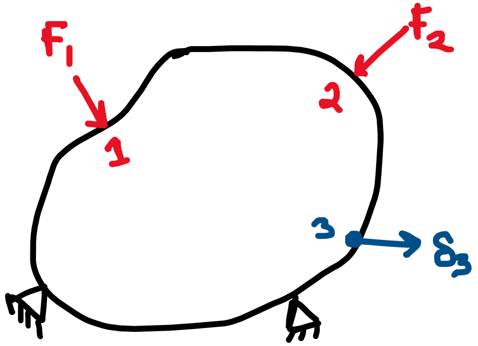
influence coefficient

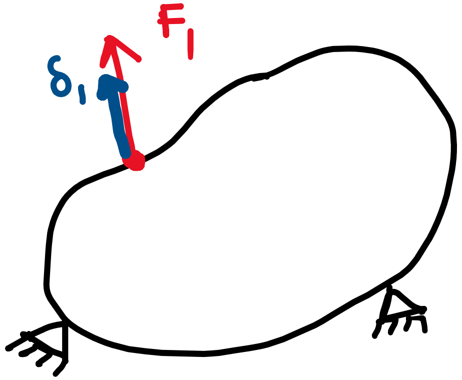
- The relationship is linear $\delta_3 \propto F_1 \;\Rightarrow\; \delta_3 = k_{31}F_1$

- We discussed some properties of influence coefficients

- We discussed superposition principle

$$\delta_3 = k_{31}F_1 + k_{32}\,F_2$$





- Then we introduced corresponding displacement (displacement at the location and in the direction of the applied force)

Objective questions

1. Which of the following holds true for influence coefficient?

   (a) it depends on the location of point where the force is being applied

   (b) it depends on the location of point where the displacement is being measured

   (c) it depends on the direction of both applied force and direction in which displacement component is being measured

   (d) it depends on the magnitude of applied force

Objective questions

2. Displacement at a point in the body is linearly related to the force applied at any other point in the body because

   (a) the equations of elasticity that we studied are linear in the unknown displacement

   (b) the boundary conditions that we studied are linear in unknown displacement

   (c) both (a) and (b)

   (d) none of these

Objective questions

3. Influence coefficient does not change even if more than one force acts in the body because

   (a) superposition principle holds

   (b) influence coefficient does change

   (c) holds only if forces applied are all parallel to each other

   (d) none of these

Objective questions

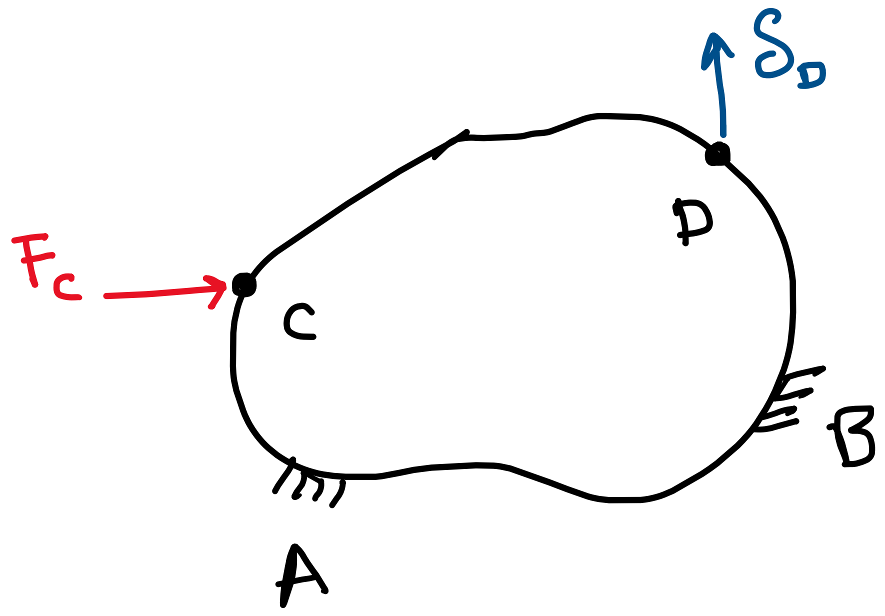
4. Suppose a body is clamped at two points (A, B) on its surface. Now a force is applied at point 'C' in x-direction and displacement is measured at another point 'D' in 'y' direction. Now suppose the body is clamped at third point 'E'.

   (a) influence coefficient for point pair (C, D) will not be affected when the body is further clamped at the third point.

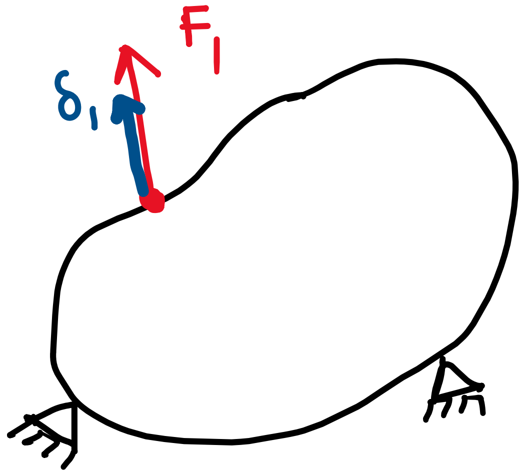
   (b) influence coefficient for point pair (C, D) will change when the body is further clamped at third point.

   (c) more information is needed

   (d) none of these



Concept of corresponding displacement



- The component of the displacement of a body at the same location and along the same direction as the applied force is called the "corresponding displacement"

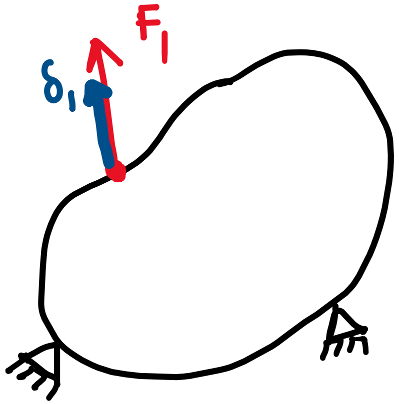
- Note that the resulting displacement vector may not point in the direction of the applied force

- The "corresponding displacement" is responsible for actual work done, since the component of displacement perpendicular to force is zero. Hence, it is also called the *work-absorbing displacement*.

- If we apply force F_1 at point 1 and measure the corresponding displacement δ_1, then they are related linearly as:

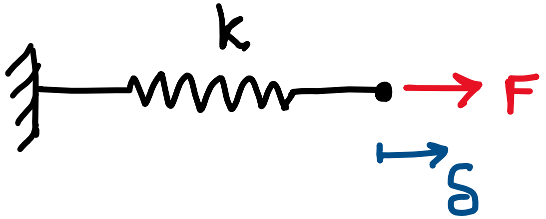
$$\delta_1 = k_{11} F_1$$

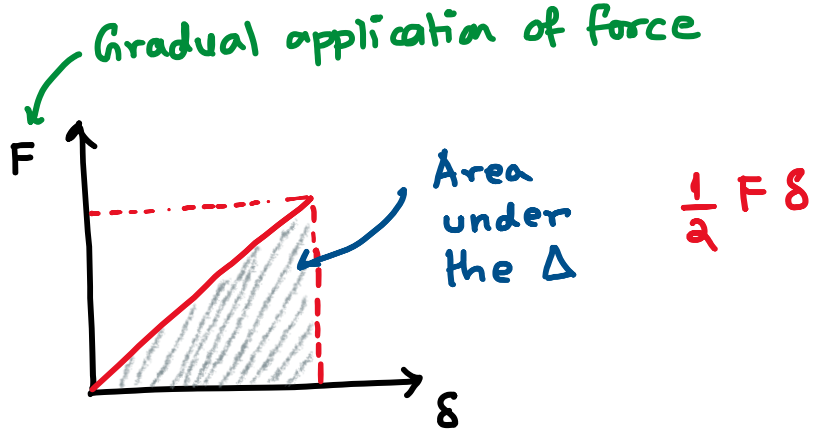
Energy stored in a deformable body due to corresponding displacement



- How much is the energy stored in a deformable body due to an applied force F_1?
- Would it be: Stored energy = $\cancel{F_1 \,.\, \delta_1}$? $\frac{1}{2} F_1 \cdot \delta_1$ (using analogy of spring)

Recall the energy stored in a spring due to <u>gradual</u> application of a force





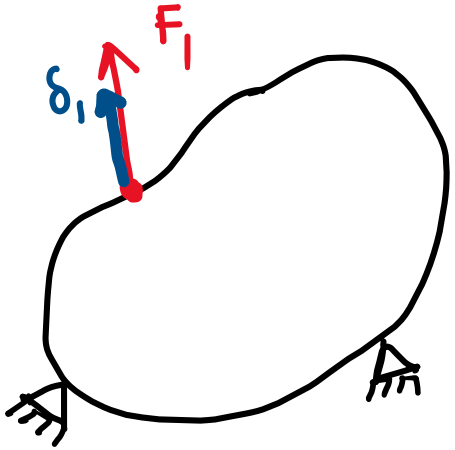
corresponding displacement

$$\text{Energy stored} = \frac{1}{2}\, k\, \delta^2 = \frac{1}{2}\,(k\delta)\,.\,\delta = \frac{1}{2} F\,.\,\delta$$

Why is the ½ here?

- Final energy stored is independent of loading path
- Even if loading is instantaneous, the energy stored in body on reaching equilibrium would still be $\frac{1}{2}\, F\,.\,\delta$

Energy stored in a deformable body due to multiple forces

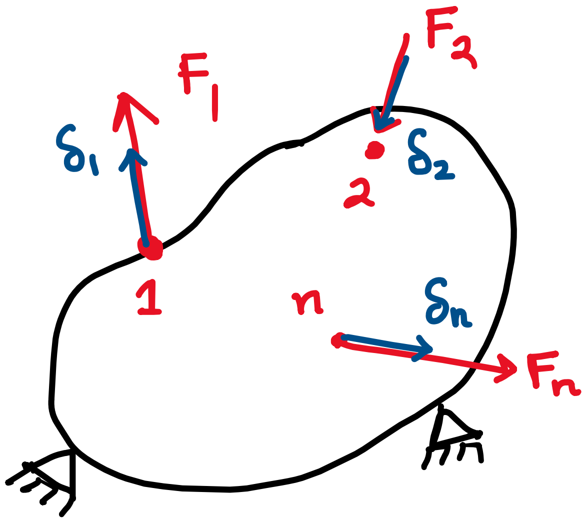


- Energy stored in a deformable body due to a single force F_1

$$\frac{1}{2}\,F_1.\,\delta_1 = \frac{1}{2}\,F_1.\,(k_{11}F_1) = \frac{1}{2}\,k_{11}F_1^2$$

- We can have multiple forces $F_1, F_2, \cdots, F_n$ acting on the body with corresponding displacements $\delta_1, \delta_2, \cdots, \delta_n$

- The total energy stored would be: $\sum_{i=1}^{n} \frac{1}{2} F_i \, \delta_i$



But, $\delta_i \neq k_{ii} F_i$

Applicable when only F_i is applied

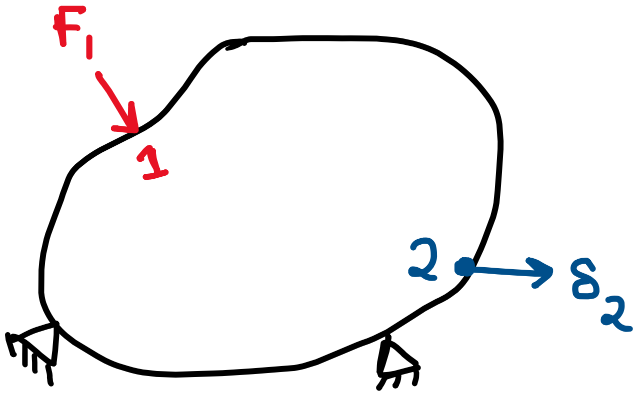
For multiple forces, use superposition

$$\delta_i = \sum_{j=1}^{n} k_{ij} F_j$$

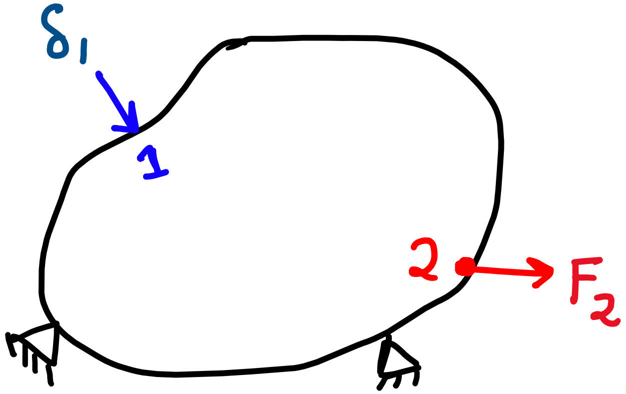
$\Rightarrow$ Total energy stored $= \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{2} k_{ij} F_i F_j$

Reciprocal relation

$\delta_2 = k_{21} F_1$



$\delta_1 = k_{12} F_2$



What is the relation between k_{12} and k_{21}?

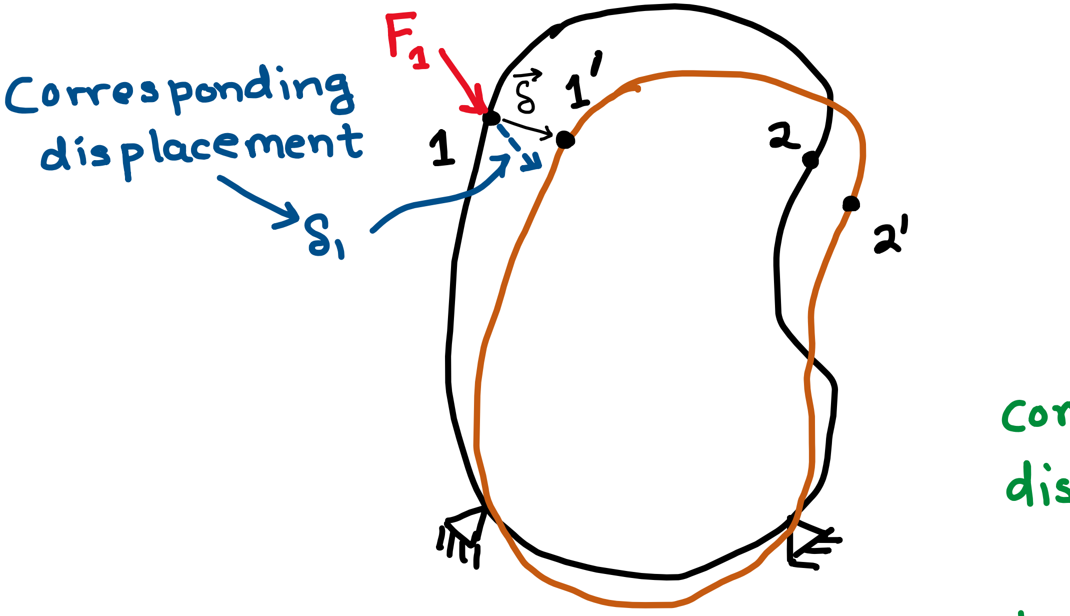
We will answer this question using energy stored in a body

Reciprocal relation

1) Lets apply F_1 on the body (and no other force)
   - Energy stored is

$$E = \frac{1}{2} F_1 \delta_1 = \frac{1}{2} F_1 (\underbrace{k_{11} \delta_1}) = \frac{1}{2} k_{11} F_1^2$$

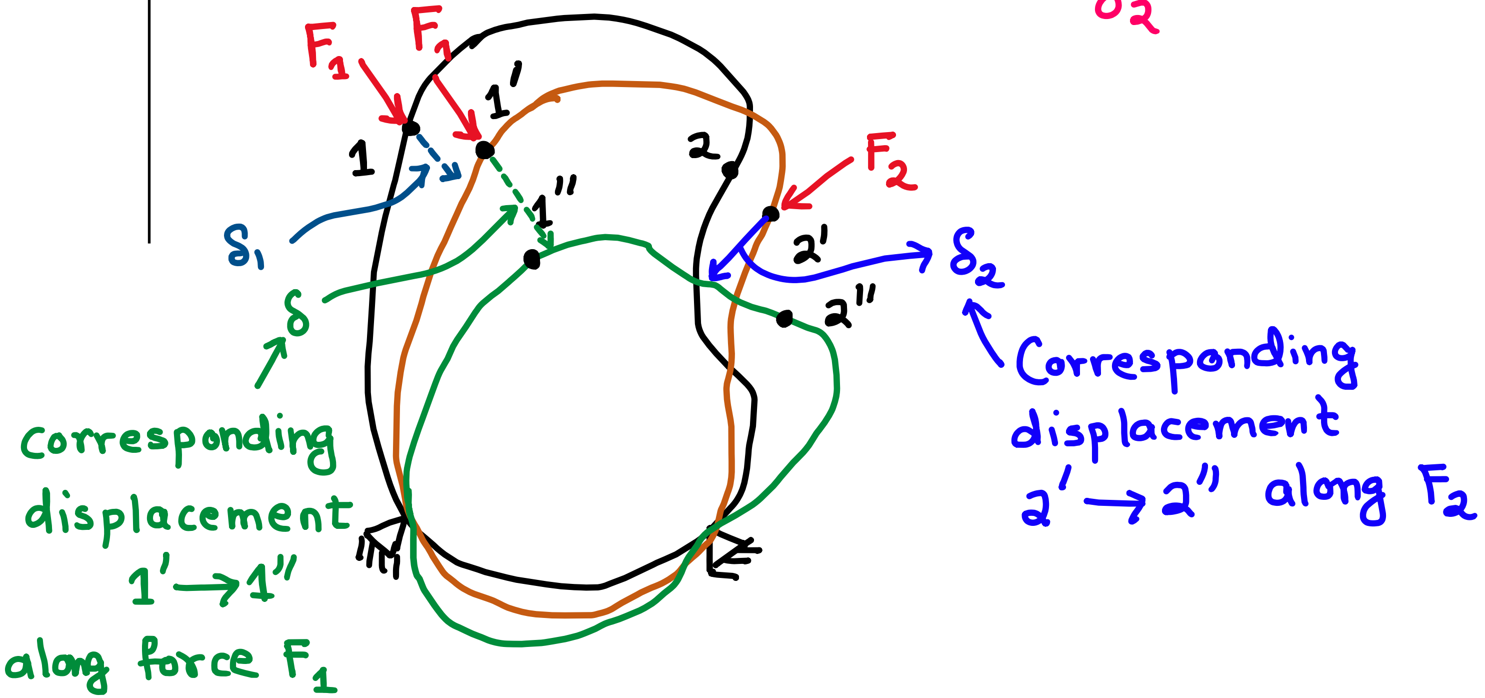
since only F_1 is acting (else superpose!)



2) Lets now apply F_2 on the body (while force F_1 acts fully)
   - Energy stored is

$$E = \underbrace{\frac{1}{2} k_{11} F_1^2}_{\text{from before}} + F_1 (\overbrace{k_{12} F_2}^{\delta}) + \frac{1}{2} F_2 (\underbrace{k_{22} F_2}_{\delta_2})$$

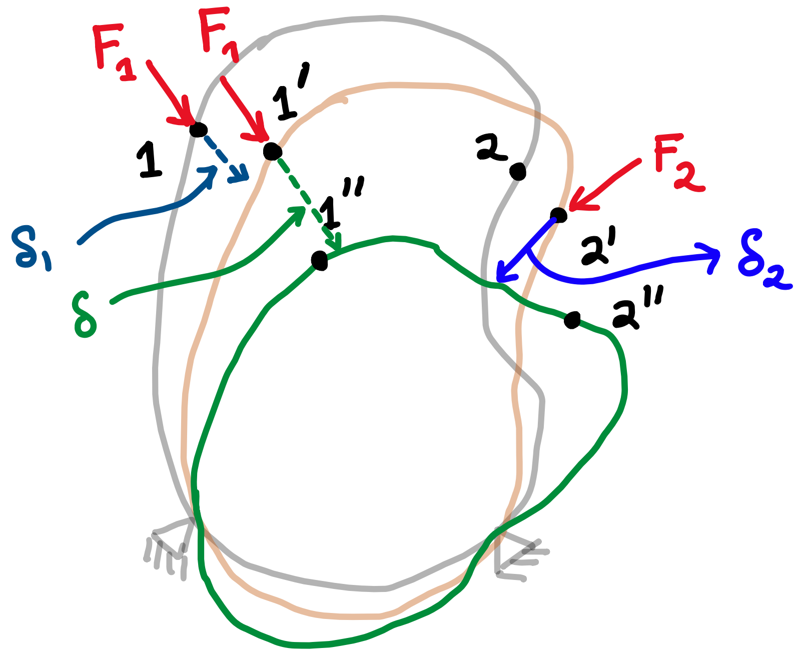
did not divide by 2 bcz F_1 was acting fully



Reciprocal relation

1) Lets apply F_1 on the body (and no other force)

2) Then apply F_2 on the body (while force F_1 acts fully)



Energy stored

$$E = \frac{1}{2} k_{11} F_1^2 + \frac{1}{2} k_{22} F_2^2 + k_{12} F_1 F_2$$

Now, think of a second process

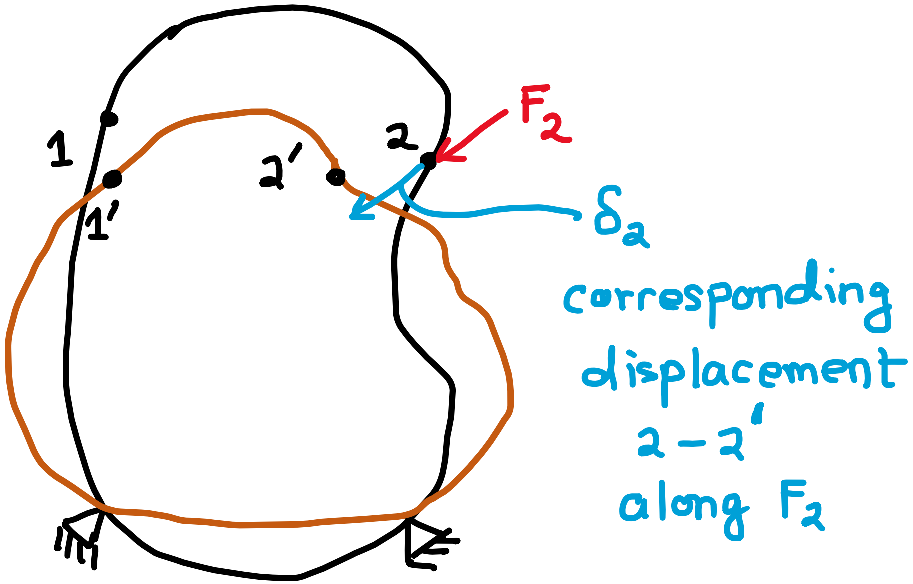
1) Lets apply F_2 on the body (and no other force)

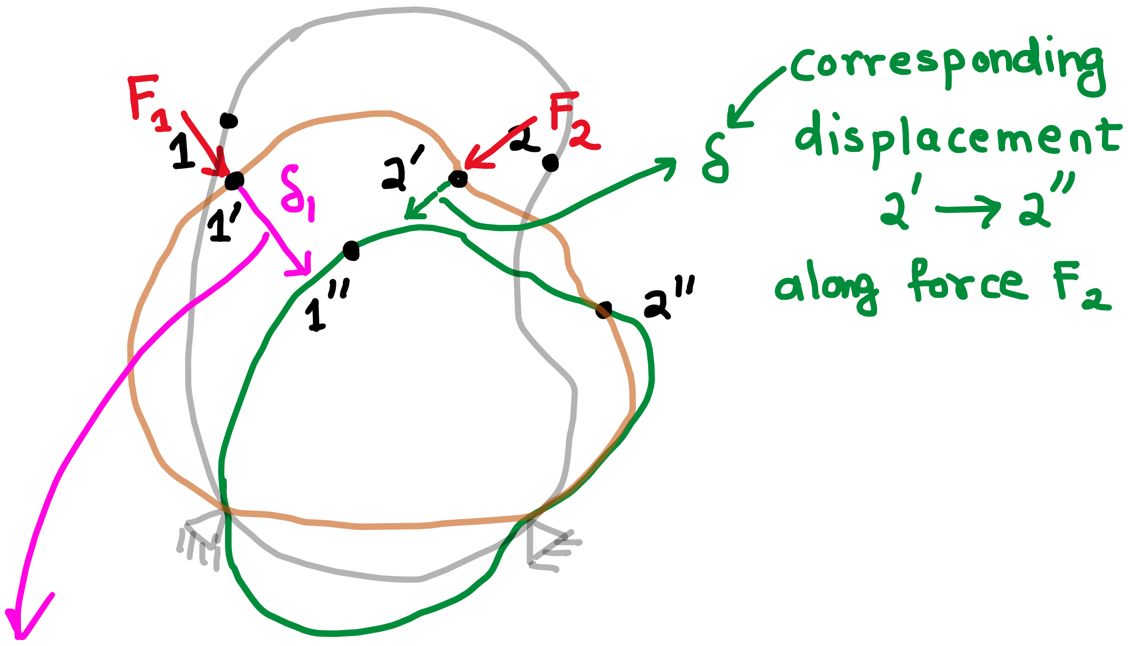
2) Then apply F_1 on the body (while force F_2 acts fully)

Reciprocal relation

1) Lets apply F_2 on the body (and no other force)

2) Then apply F_1 on the body (while force F_2 acts fully)





$$E = \frac{1}{2} F_2 \delta_2 = \frac{1}{2} \underbrace{k_{22} F_2^2}$$

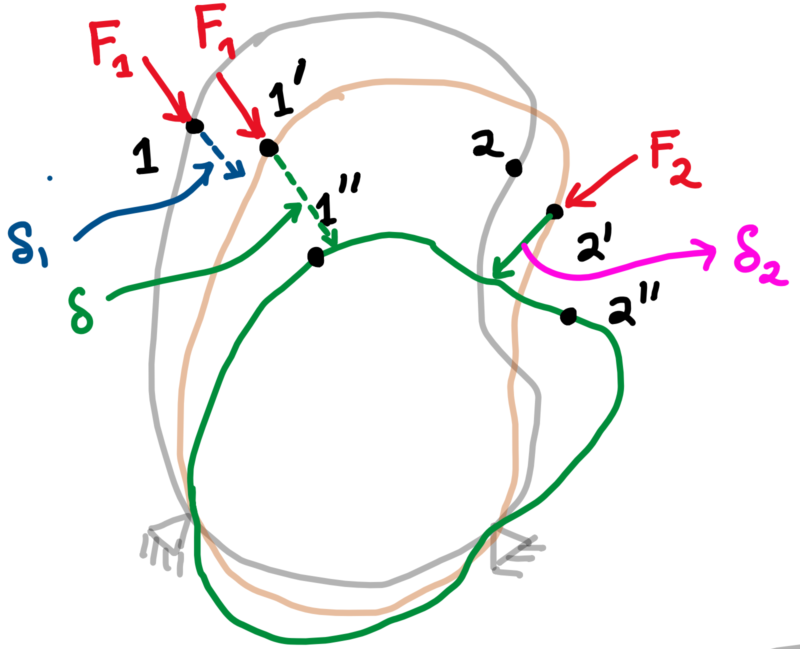
$$E = \frac{1}{2} k_{22} F_2^2 + \left(\frac{1}{2} F_1 \overbrace{(k_{11} F_1)}^{\delta_1} + F_2 \overbrace{(k_{21} F_1)}^{\delta} \right)$$

$$E = \frac{1}{2} k_{22} F_2^2 + \frac{1}{2} k_{11} F_1^2 + k_{21} F_1 F_2$$

Reciprocal relation

1) Lets apply F_1 on the body (and no other force)

2) Then apply F_2 on the body (while force F_1 acts fully)

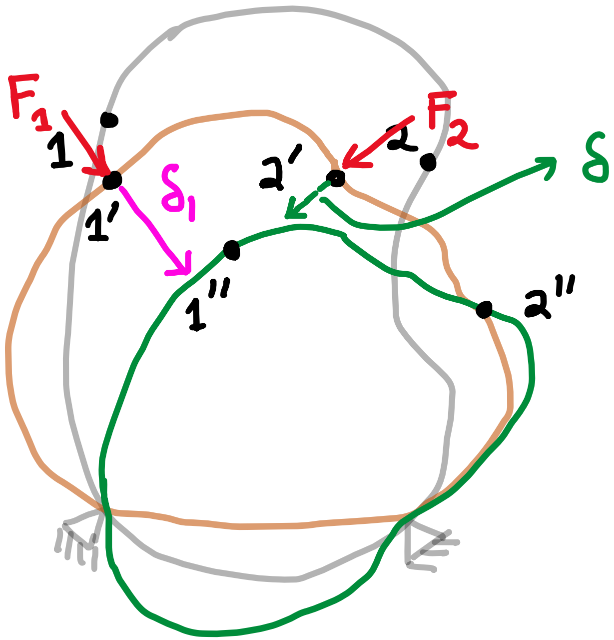


Energy stored

$$E = \frac{1}{2} k_{11} F_1^2 + \frac{1}{2} k_{22} F_2^2 + k_{12} F_1 F_2$$

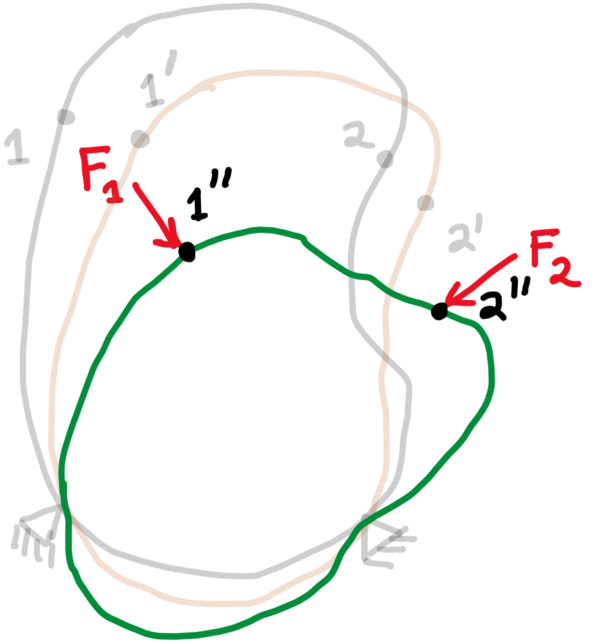
1) Lets apply F_2 on the body (and no other force)

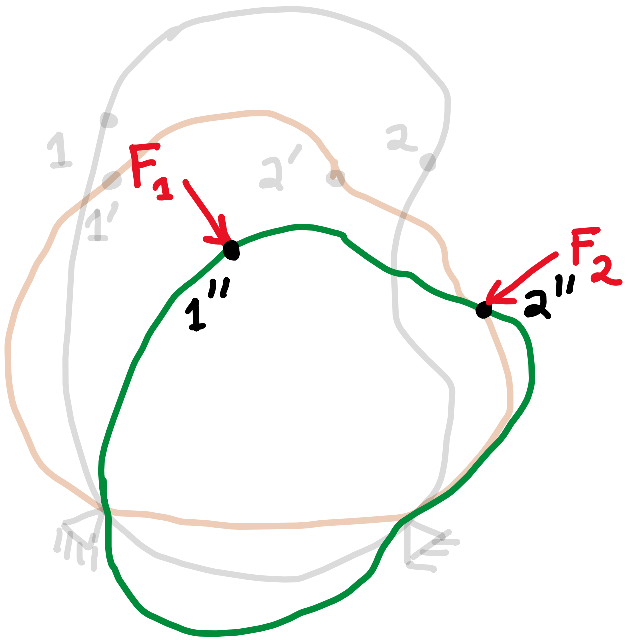
2) Then apply F_1 on the body (while force F_2 acts fully)



$$E = \frac{1}{2} k_{22} F_2^2 + \frac{1}{2} k_{11} F_1^2 + k_{21} F_1 F_2$$

Reciprocal relation





$$E = \frac{1}{2}k_{11}F_1^2 + \frac{1}{2}k_{22}F_2^2 + k_{12}F_1F_2$$

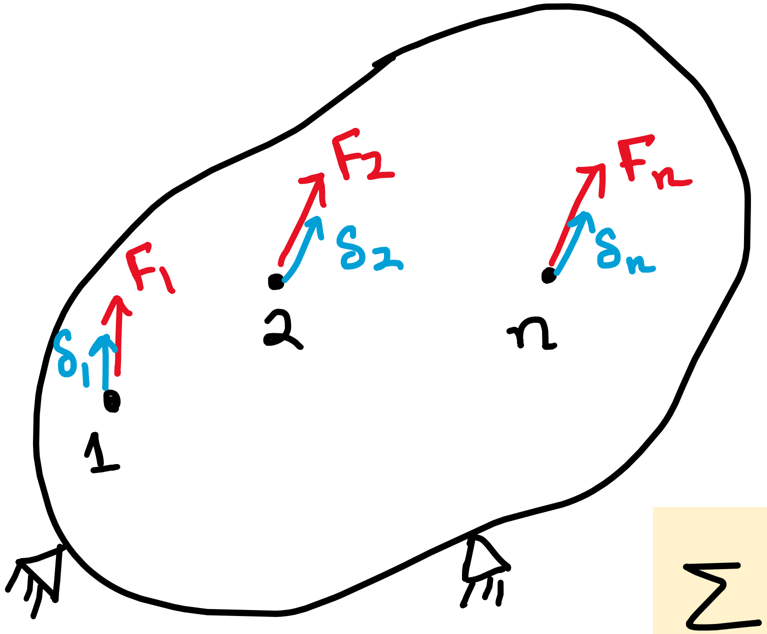
$$E = \frac{1}{2}k_{22}F_2^2 + \frac{1}{2}k_{11}F_1^2 + k_{21}F_1F_2$$

- Now the final states are the same and it has both F_1 and F_2 acting in the final state
- Therefore, the energy stored must also be the same. If we compare the two energies, we see that

$$\boxed{k_{12} = k_{21}}$$

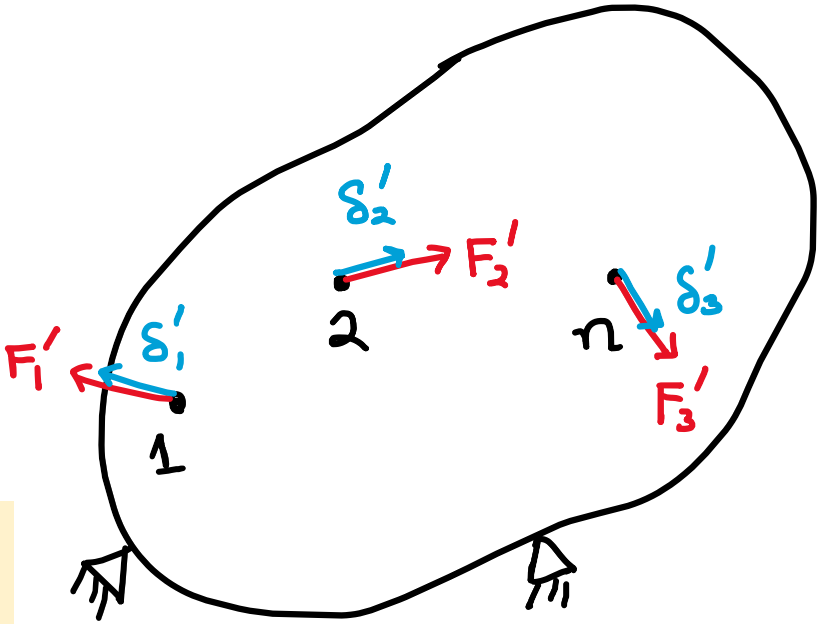
Maxwell-Betti-Rayleigh reciprocal theorem

We worked out the reciprocal relation in previous slide, now we look at the RECIPROCAL theorem



Situation 1

$$\sum_i F_i \, \delta_i' = \sum_i F_i' \, \delta_i$$



Situation 2

Forces $\rightarrow$ $F_1, F_2, \dots, F_n$

Corresponding displacements $\rightarrow$ $\delta_1, \delta_2, \dots, \delta_n$

Forces $\rightarrow$ $F_1', F_2', \dots, F_n'$

Corresponding displacements $\rightarrow$ $\delta_1', \delta_2', \dots, \delta_n'$

Theorem: The work done by forces in situation 1 through the corresponding displacements of situation 2 equals the work done by forces in situation 2 through the corresponding displacements of situation 1

Proof of Maxwell-Betti-Rayleigh reciprocal theorem

$$\sum_i F_i \delta_i' = \sum_i F_i' \delta_i$$

Reciprocal relation

$$k_{ij} = k_{ji}$$

$$\sum_i F_i \delta_i' = \sum_i F_i \left(\sum_j k_{ij} F_j'\right) = \sum_i F_i' \delta_i$$

$$\begin{aligned}
= \; & F_1 (k_{11} F_1' + k_{12} F_2' + \cdots + k_{1n} F_n') \\
+ \; & F_2 (k_{21} F_1' + k_{22} F_2' + \cdots + k_{2n} F_n') \\
+ \; & \cdots \\
& \vdots \\
+ \; & F_n (k_{n1} F_1' + k_{n2} F_2' + \cdots + k_{nn} F_n')
\end{aligned}$$

$$F_1' (k_{11} F_1 + k_{21} F_2 + \cdots + k_{n1} F_n)$$

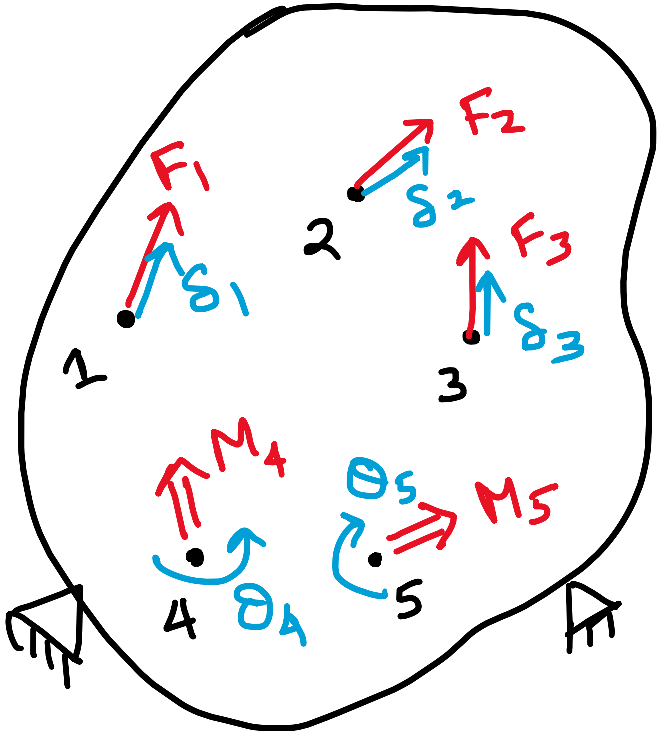
$$= F_1' (k_{11} F_1 + k_{12} F_2 + \cdots + k_{1n} F_n)$$

$$= F_1' \delta_1$$

$$F_2' (k_{12} F_1 + k_{22} F_2 + \cdots + k_{n2} F_n)$$

$$= F_2' \delta_2$$

Generalized forces and generalized displacements



Moments : Generalized forces

Rotations (θ) : Generalized displacements

$$\theta_4 = k_{41} F_1 + k_{42} F_2 + k_{43} F_3 + k_{44} M_4 + k_{45} M_5$$

Units: θ/force $\quad$ θ/moment

$$\delta_1 = k_{11} F_1 + k_{12} F_2 + k_{13} F_3 + k_{14} M_4 + k_{15} M_5$$

$\frac{\text{disp}}{\text{force}}$ $\quad$ $\frac{\text{disp}}{\text{moment}} = \frac{\text{rot}}{\text{force}}$