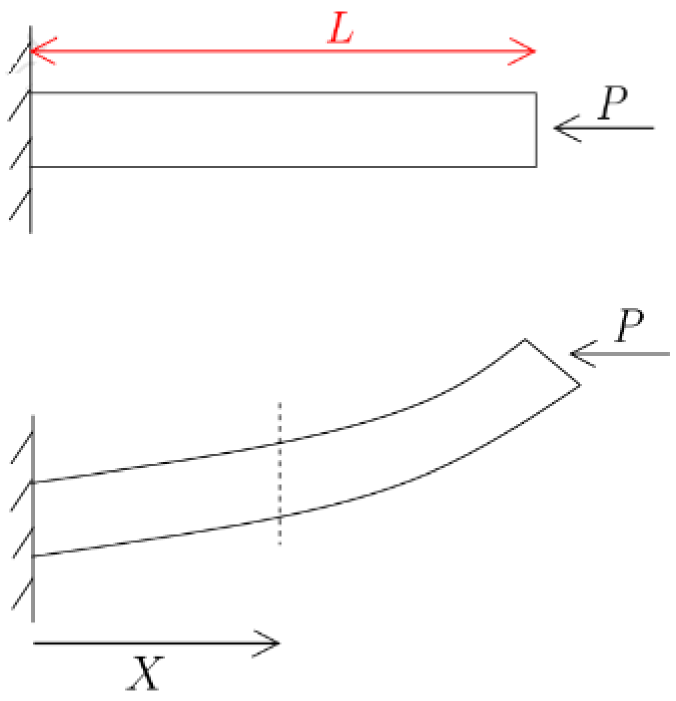
# Classroom Lecture 33

Buckling of beams (contd…)

Introduction to Energy methods

# Buckling of beams

# Beam buckling (contd from last lecture)




- When the compressive load $P$ reaches the critical value, the beam/column will bend, even though we are applying axial compressive load → **Buckling** phenomena

- The minimum value of the axial force at which the beam will buckle is the called the critical buckling load, $P_{cr}$

- For the cantilever beam, $P_{cr}$ was derived as:

$$P^{cr} = \frac{\pi^2 EI}{4L^2}$$

- If the compressive axial force is less than $P_{cr}$, the beam will remain straight

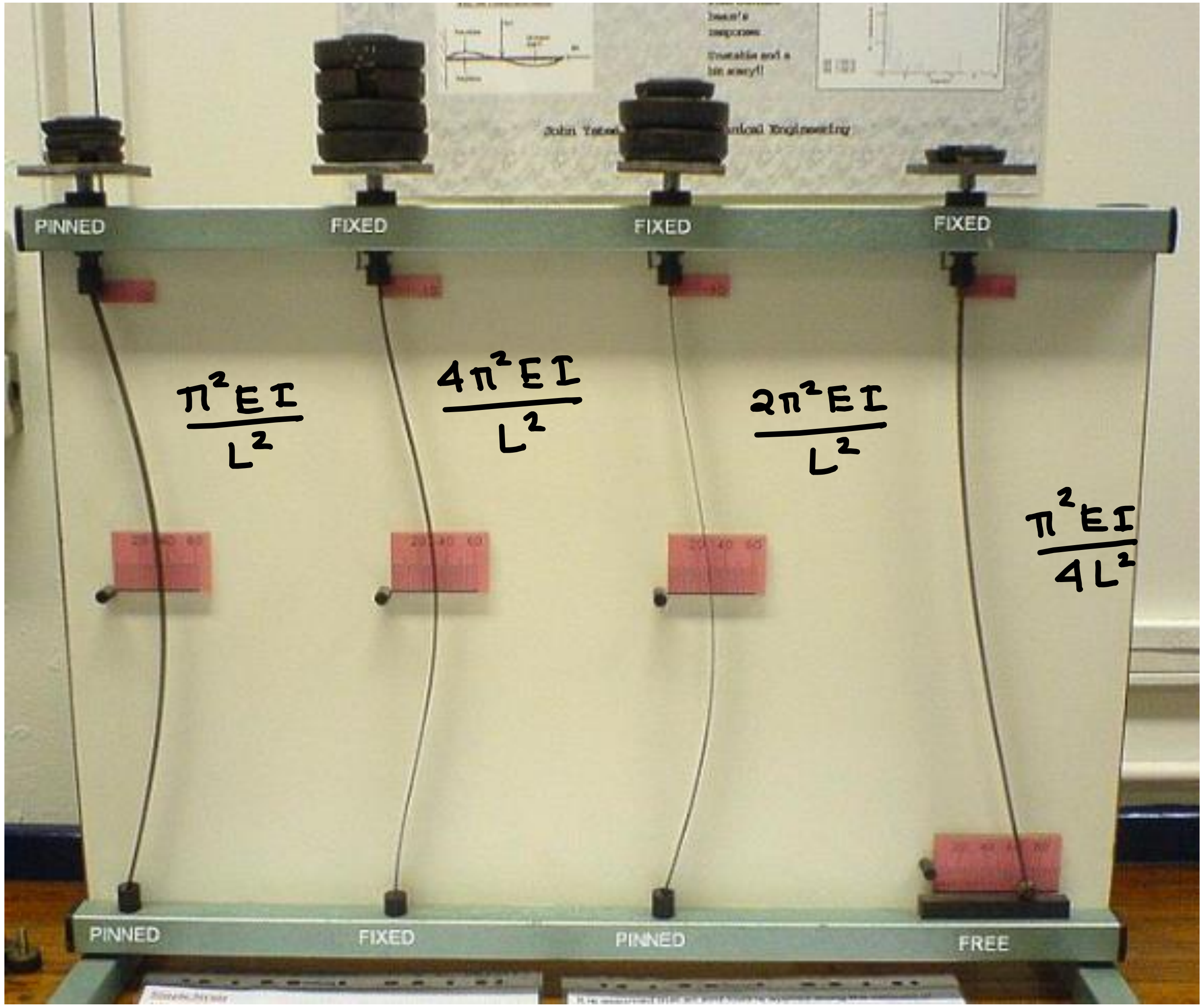
- $P_{cr}$ is proportional to bending stiffness EI and inversely proportional to square of the beam's length
  - Longer beam requires lesser force to buckle

# Beam buckling depends upon the boundary conditions

- In class, $P_{cr}$ was derived for cantilever beam, using the relevant boundary conditions

- The critical buckling load also depends upon the boundary conditions of the beam/column

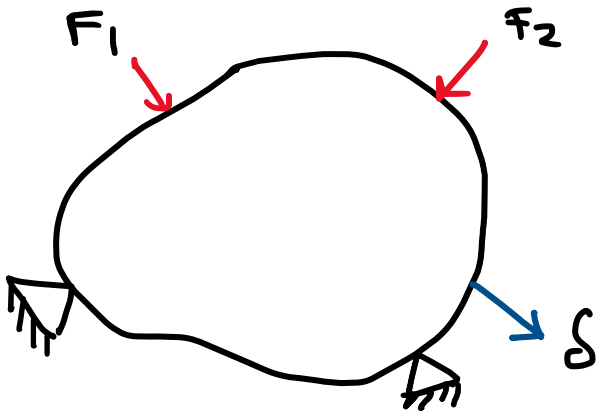
- Some typical boundary conditions encountered are:
  - Pinned-pinned
  - Fixed-fixed
  - Fixed-pinned
  - Fixed-free (i.e. cantilever)



# Introduction to Energy Methods

# How did we solve for displacements of a body given applied forces?

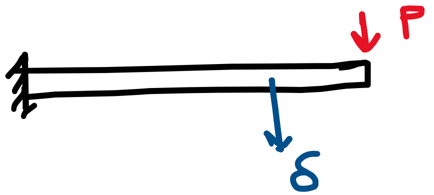
- We have solved equations of 3D-elasticity in the form of PDEs to obtain deformation of a body




$$\begin{aligned}
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + b_x &= \rho a_x \\
\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + b_y &= \rho a_y \\
\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + b_z &= \rho a_z
\end{aligned}$$

$$\begin{aligned}
\underline{\underline{\sigma}}\,\underline{n} &= \underline{t}_0, \quad \text{(traction boundary condition)} \\
\underline{u} &= \underline{u}_0. \quad \text{(displacement boundary condition)}
\end{aligned}$$

- In last two lectures, we solved ODEs to solve beam deflection problems



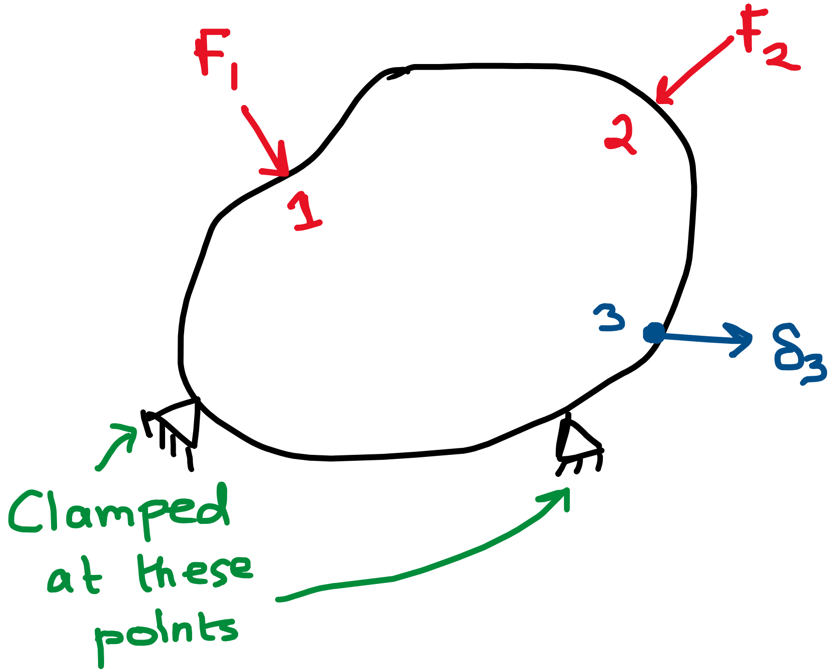

$$EI\frac{d^2y}{dX^2} = M(X) \qquad EI\frac{d\theta}{dX} = M(X) \quad \frac{dy}{dX} - \theta = \frac{V}{kGA}$$

- There is also another approach of finding deformation, using energy methods!

Recall that you could solve motion of rigid bodies using either Newton's 2nd law or using minimization of energy
For many complex problems, it becomes easier to find deformations using energy methods!

# **Concepts** needed for energy methods

# How is the displacement at a point related to a force at another point?




- Consider a body subjected to (concentrated) forces at point 1 and point 2, and we are interested in measuring the displacement at point 3
- What is the relationship between the displacement $\delta_3$ at point 3 and the force $F_1$ at point 1?

$$\delta_3 \equiv F_1$$

- Is the relationship linear?
- If linear, what would be proportionality constant?

- To answer these questions, let's have a look at the 3D equations of elasticity

# Linearity of the elasticity equations and boundary conditions

- 3D equations of elasticity (a set of PDEs)

$$\begin{cases} \dfrac{\partial \sigma_{xx}}{\partial x} + \dfrac{\partial \tau_{xy}}{\partial y} + \dfrac{\partial \tau_{xz}}{\partial z} + b_x = \rho a_x \\ \dfrac{\partial \tau_{yx}}{\partial x} + \dfrac{\partial \sigma_{yy}}{\partial y} + \dfrac{\partial \tau_{yz}}{\partial z} + b_y = \rho a_y \\ \dfrac{\partial \tau_{zx}}{\partial x} + \dfrac{\partial \tau_{zy}}{\partial y} + \dfrac{\partial \sigma_{zz}}{\partial z} + b_z = \rho a_z \end{cases}$$

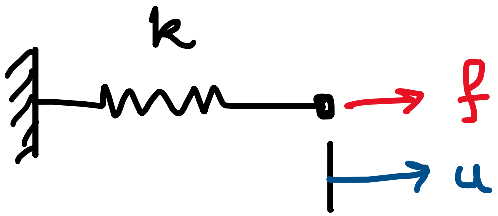
$$\begin{aligned} \underline{\underline{\sigma}}\,\underline{n} &= \underline{t}_0, && \text{(traction boundary condition)} \\ \underline{u} &= \underline{u}_0. && \text{(displacement boundary condition)} \end{aligned}$$

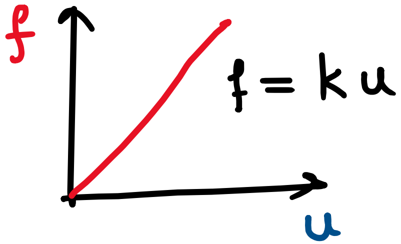
- The relation between stress and displacement is linear! **How?**
  - These equations are linear in stress components $\sigma$ and $\tau$ (e.g. power of stress is one)
  - Stress is linear in strain $\sigma_{ij} = \lambda\, tr\left(\underline{\underline{\epsilon}}\right)\delta_{ij} + 2\mu\epsilon_{ij}$
  - Strain is linear in displacement $\underline{\underline{\epsilon}} = \frac{1}{2}\left[\nabla \underline{u} + (\nabla \underline{u})^T\right]$
  - Boundary conditions are also linear in displacement $\underline{u}$

So elasticity equations and the boundary conditions are all linear in the unknown displacement

# What does linearity in equations imply?

- Ex 1: Linear spring
  - $u$ is unknown static disp, $f$ is known static force
  - $k\,u = f$
  - Double the force $f$, you get double $u$




If equation is linear, the unknown displacement is linear function of the force applied

- If you apply $f_1(t) \rightarrow$ you get $u_1(t)$
- If you apply $f_2(t) \rightarrow$ you get $u_2(t)$
- If you apply $f_1(t) + f_2(t) \rightarrow$ you get $u_1(t) + u_2(t)$

Superposition principle holds true for linear systems!

# Linearity and superposition apply to 3D linear elasticity equations as well

- Here, we have a set of PDEs, and they are still **linear**!

$$\begin{aligned}
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + b_x &= \rho a_x \\
\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + b_y &= \rho a_y \\
\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + b_z &= \rho a_z
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$$\begin{aligned}
\underline{\underline{\sigma}}\,\underline{n} &= \underline{t}_0, \quad \text{(traction boundary condition)} \\
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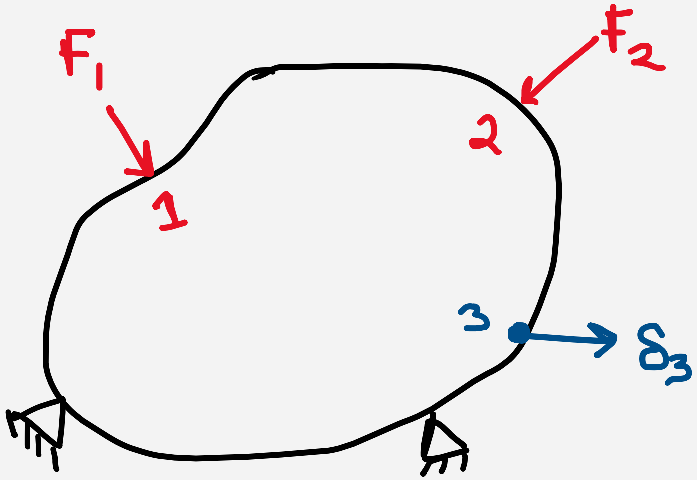
\end{aligned}$$

- What the forces that act here?
  - Body forces: $b_x, b_y, b_z$
  - Surface tractions: $\underline{t}_0$ (through boundary conditions)
- If you double these forces, the resulting deformation will also get doubled!

# How is the displacement at a point related to a force at another point?




- Consider a body subjected to (concentrated) forces at point 1 and point 2, and we are interested in measuring the displacement at point 3
- What is the relationship between the displacement $\delta_3$ at point 3 and the force $F_1$ at point 1?

$$\delta_3 \equiv F_1$$

- Is the relationship linear?
- If linear, what would be proportionality constant?

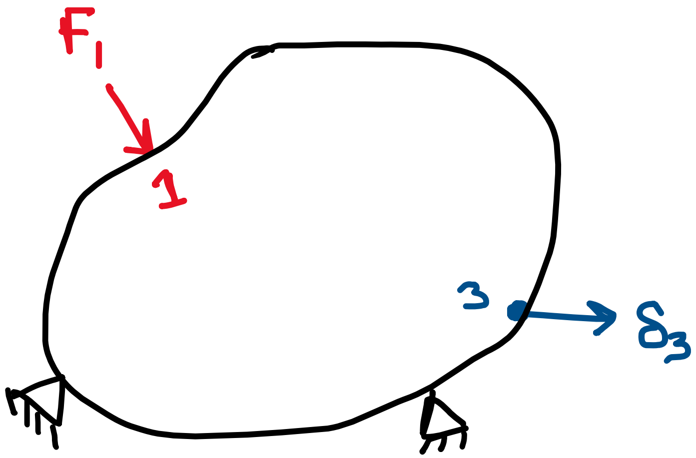
- Since the unknown displacement is **linearly related** to the force, we can write

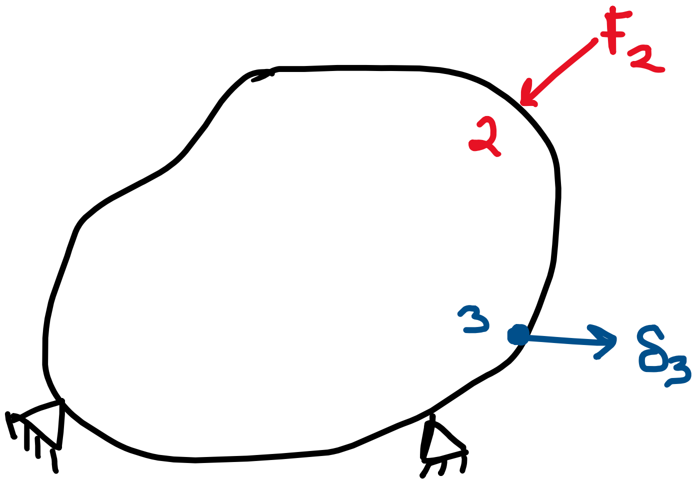
$$\delta_3 \propto F_1 \;\Rightarrow\; \delta_3 = k_{31} F_1$$

- $k_{31}$ is the proportionality constant and is called the **influence coefficient**

# Properties of influence coefficients

- Influence coefficients vary depending upon the **location of the applied force**




$$\delta_3 = k_{31} F_1 \qquad \delta_3 = k_{32} F_2$$

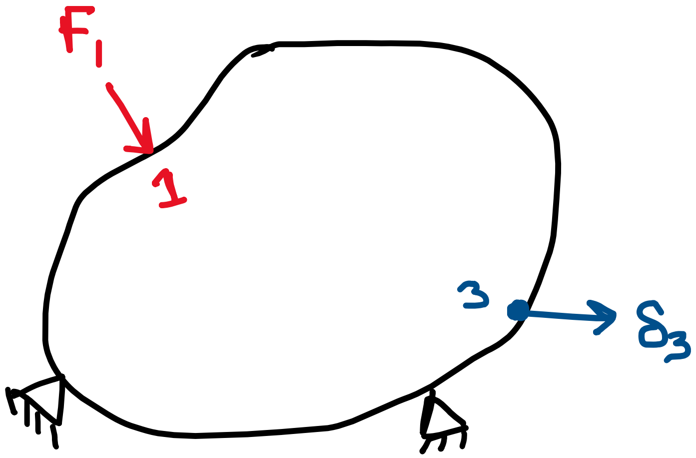
$$k_{31} \neq k_{32}$$

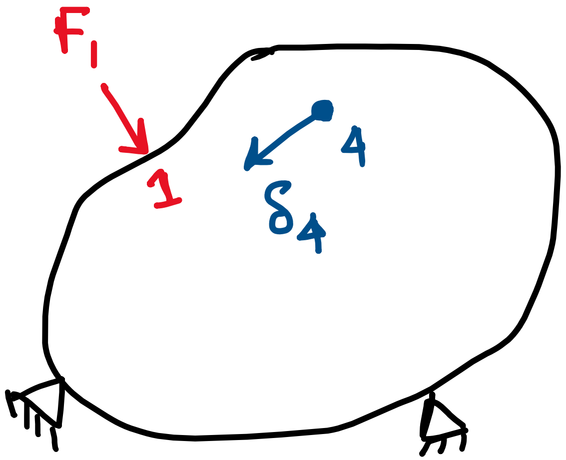
Influence coefficients depend on

- The **location of the applied force**

# Properties of influence coefficients

- Influence coefficients vary depending upon the **location of the measured displacement**




$$\delta_3 = k_{31} F_1$$

$$\delta_4 = k_{41} F_1$$

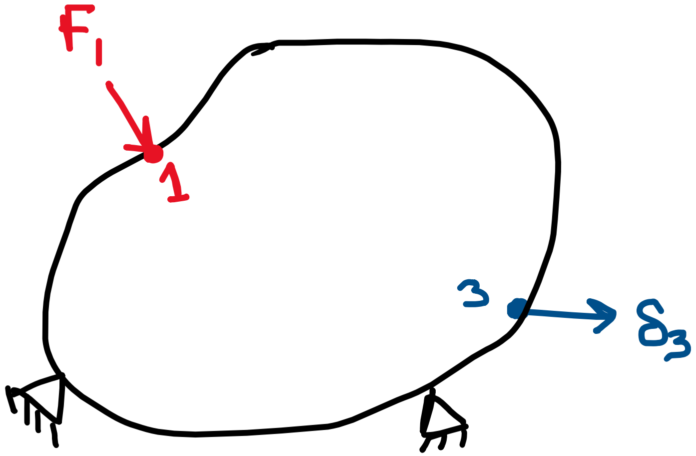
$$k_{31} \neq k_{41}$$

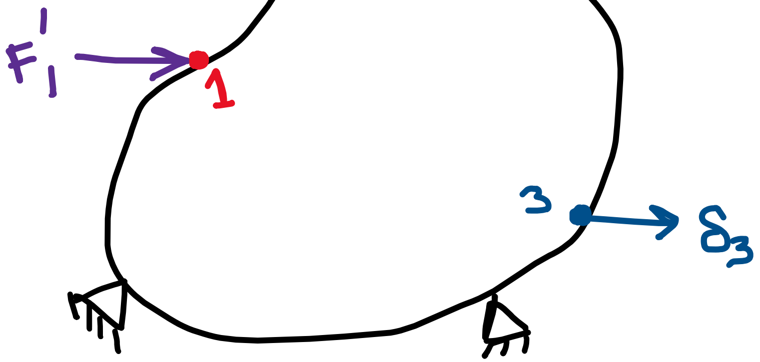
Influence coefficients depend on

- The **location of the applied force** and the **location of the the measured displacement**

# Properties of influence coefficients

- Influence coefficients depend on the **direction of the applied force**




$$\delta_3 = k_{31} F_1$$

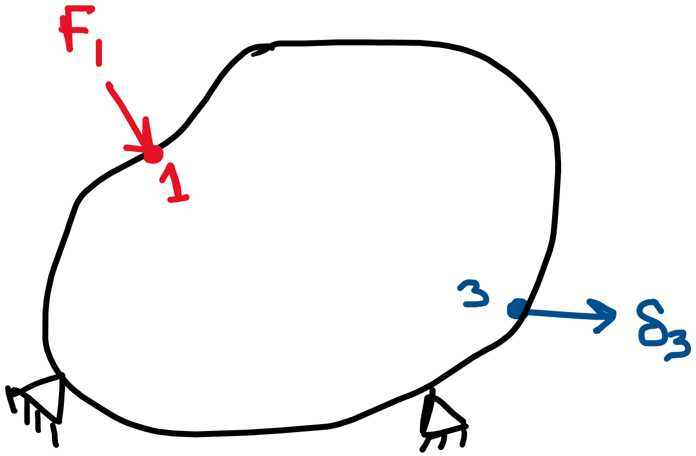
$$\delta_3 = k_{31}' F_1'$$

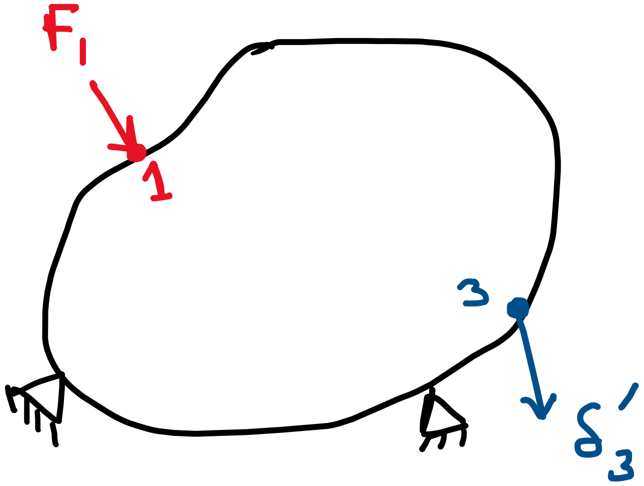
$$k_{31} \neq k_{31}'$$

Influence coefficients depend on
- The **location of the applied force** and the **location of the the measured displacement**
- The **direction of the applied force** but not on the value of the force!

# Properties of influence coefficients

- Influence coefficients depend on the **direction of the measured displacement**




$$\delta_3 = k_{31} F_1$$

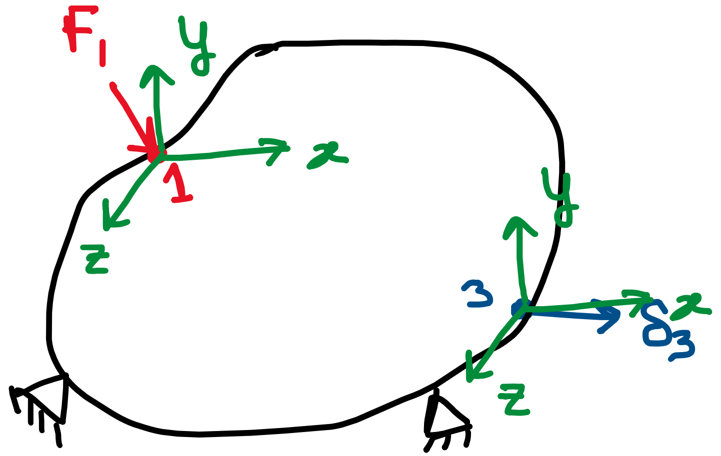
$$\delta_3' = k_{31}' F_1$$

$$k_{31} \neq k_{31}'$$

Influence coefficients depend on
- The **location of the applied force** and the **location of the the measured displacement**
- The **direction of the applied force** but not on the value of the force!
- The **direction in which the displacement is measured**

# Properties of influence coefficients

- Influence coefficients are direction-dependent
- Both displacement and force are vector quantities
- Each have three components in three mutually perpendicular directions

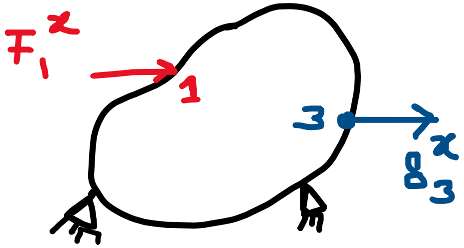



$$F_1 = \begin{Bmatrix} F_1^x \\ F_1^y \\ F_1^z \end{Bmatrix}$$

Component of force at point 1 in x-direction

$$\delta_3 = \begin{Bmatrix} \delta_3^x \\ \delta_3^y \\ \delta_3^z \end{Bmatrix}$$

Component of displacement at point 3 in x-direction




$$\delta_3^x = k_{31}^{xx} F_1^x \qquad \delta_3^y = k_{31}^{yx} F_1^x \qquad \delta_3^z = k_{31}^{zx} F_1^x$$

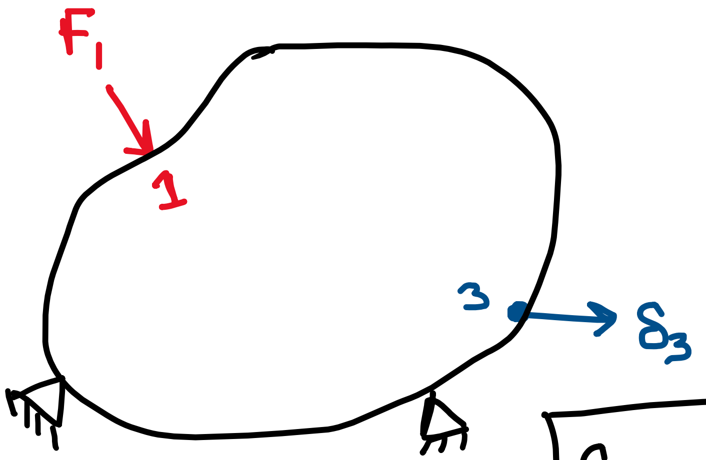
$$\delta_3^x = k_{31}^{xy} F_1^y \qquad \delta_3^y = k_{31}^{yy} F_1^y \qquad \delta_3^z = k_{31}^{zy} F_1^y$$

$$\delta_3^x = k_{31}^{xz} F_1^z \qquad \delta_3^y = k_{31}^{yz} F_1^z \qquad \delta_3^z = k_{31}^{zz} F_1^z$$

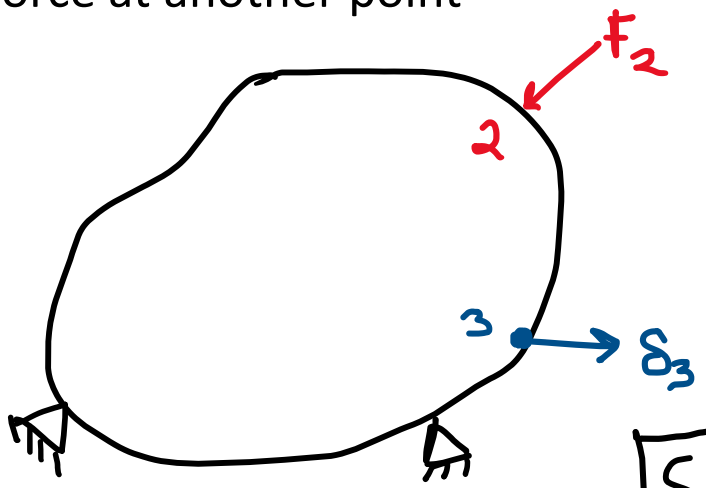
Each equation resemble the Hooke’s law (where stress was linearly related to strain)

# Displacement expressed as superposition using influence coefficients

- Displacement at one point is a linear function of applied force at another point



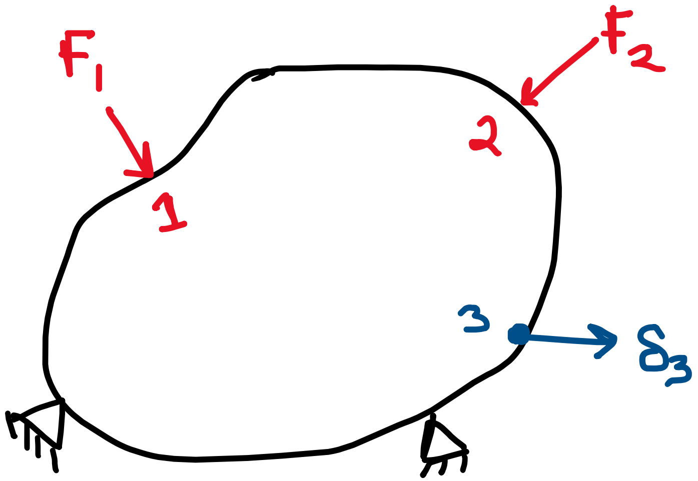

$$\boxed{\delta_3 = k_{31} F_1}$$



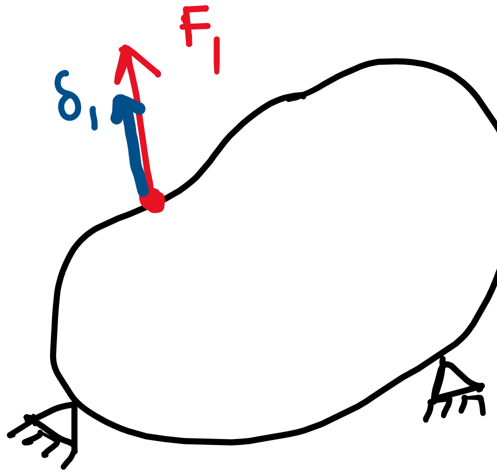

$$\boxed{\delta_3 = k_{32} F_2}$$

- Due to superposition, displacement at a point due to forces applied at two different points can be written using superposition principle

$$\boxed{\delta_3 = k_{31} F_1 + k_{32} F_2}$$



# Concept of corresponding displacement




- The component of the displacement of a body at the same location and along the same direction as the applied force is called the "corresponding displacement"

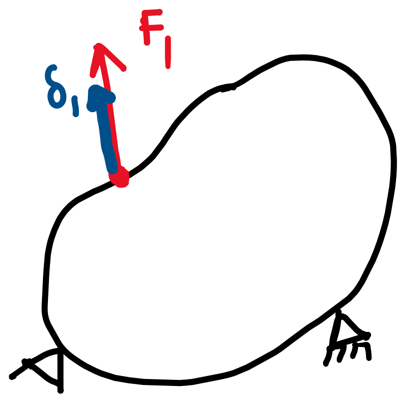
- Note that the resulting displacement vector may not point in the direction of the applied force

- The "corresponding displacement" is responsible for actual work done, since the component of displacement perpendicular to force is zero. Hence, it is also called the *work-absorbing displacement*.

- If we apply force $F_1$ at point 1 and measure the corresponding displacement $\delta_1$, then they are related linearly as:

$$\delta_1 = k_{11}\, F_1$$

# Energy stored in a deformable body due to corresponding displacement




- How much is the energy stored in a deformable body due to an applied force $F_1$?

- Would it be:  Stored energy = $F_1 \, . \, \delta_1$ ?