Second order tensors

Note Title 8/6/2022

$\underline{\underline{C}} = \sum_i \sum_j C_{ij} \, \underline{e}_i \otimes \underline{e}_j$ $\qquad v_i = \underbrace{\underline{v} \cdot \underline{e}_i}$

$C_{mn} = (\underline{\underline{C}} \cdot \underline{e}_n) \cdot \underline{e}_j$

<u>Multiplication of a 2nd order tensor with a vector</u>
(Contraction)

$\underline{c} = \underline{\underline{A}}\underline{b} = \underline{\underline{A}} \cdot \underline{b}$ $\qquad \underline{\underline{A}} \otimes \underline{b}$, $\quad \underline{\underline{A}} \cdot \underline{b}$ $\quad \cancel{\underline{\underline{A}} \times \underline{b}}$

$$= \left(\sum_i \sum_j A_{ij} \, \underline{e}_i \otimes \underline{e}_j\right) \cdot \left(\sum_k b_k \underline{e}_k\right)$$

$$= \sum_i \sum_j \sum_k A_{ij} b_k \, (\underline{e}_i \otimes \underline{e}_j) \cdot \underline{e}_k$$

$\underline{e}_j \cdot \underline{e}_k = \begin{cases} 1 & (j=k=1) \\ 0 & (j=1, k=2) \end{cases}$

$$= \sum_i \sum_j \sum_k A_{ij} b_k \, \underline{e}_i \, \delta_{jk}$$

$\delta_{jk} = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k \end{cases}$

$$= \sum_i \sum_j A_{ij} b_j \, \underline{e}_i \qquad = \sum_i \sum_k A_{ik} b_k \underline{e}_i$$

$$\underline{c} = \sum_i \underbrace{\left(\sum_j A_{ij} b_j\right)}_{m_i} \underline{e}_i = \sum_i m_i \underline{e}_i$$

$$\Rightarrow \quad c_i = m_i = \sum_j A_{ij} b_j$$

$$c_i = \begin{bmatrix} A_{i1} & A_{i2} & A_{i3} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$\underline{c} = \underline{\underline{A}} \cdot \underline{b}$

$$\Rightarrow \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} & & \\ & \underline{\underline{A}} & \\ & & \end{bmatrix} \begin{bmatrix} \underline{b} \end{bmatrix}$$

$$C_{mn} = (\underline{\underline{C}} \cdot \underline{e}_n) \cdot \underline{e}_m$$

$$= \left[\left(\sum_i \sum_j C_{ij} \, \underline{e}_i \otimes \underline{e}_j\right) \cdot \underline{e}_n\right] \cdot \underline{e}_m$$

$$= \left[\sum_i \sum_j C_{ij} \, \underline{e}_i \, \delta_{jn}\right] \cdot \underline{e}_m$$

$$= \sum_i \sum_j C_{ij} \, \delta_{jn} \delta_{im} = C_{mn}$$

$(\underline{\underline{C}} \cdot \underline{e}_n) \cdot \underline{e}_m$

$(\underline{\underline{C}} \cdot \underline{e}_3) \cdot \underline{e}_1 = C_{13}$

$$\left(\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} C_{13} \\ C_{23} \\ C_{33} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = C_{13}$$

$$\underline{\underline{C}} = \underline{a} \otimes \underline{b}$$

$$C_{ij} = \left[(\underline{a} \otimes \underline{b}) \cdot \underline{e}_j \right] \cdot \underline{e}_i$$

$$= (\underline{b} \cdot \underline{e}_j)(\underline{a} \cdot \underline{e}_i) = a_i b_j$$

$$\underline{\underline{C}} = \underline{a} \otimes \underline{\underline{D}}\, \underline{b}$$

Multiplying two tensors.

$$\underline{\underline{C}} \cdot \underline{\underline{B}} = \underline{\underline{A}} \Longrightarrow \underline{\underline{C}} \cdot \underline{\underline{B}}$$

$$\left(\sum_i \sum_j C_{ij}\, \underline{e}_i \otimes \underline{e}_j \right) \cdot \left(\sum_k \sum_l B_{kl}\, \underline{e}_k \otimes \underline{e}_l \right)$$

$$= \sum_i \sum_j \sum_k \sum_l C_{ij} B_{kl} \delta_{jk}\, \underline{e}_i \otimes \underline{e}_l$$

$$= \sum_i \sum_j \sum_l C_{ij} B_{jl}\, \underline{e}_i \otimes \underline{e}_l$$

$$\underline{\underline{A}} = \sum_i \sum_l \underbrace{\left(\sum_j C_{ij} B_{jl} \right)}_{A_{il}} \underline{e}_i \otimes \underline{e}_l$$

$$A_{il} = \sum_j C_{ij} B_{jl}$$

$$= \begin{bmatrix} C_{i1} & C_{i2} & C_{i3} \end{bmatrix} \begin{bmatrix} B_{1l} \\ B_{2l} \\ B_{3l} \end{bmatrix} \rightarrow l^{th} \text{ column}$$

i^{th} rows

$$\Rightarrow [\underline{\underline{A}}] = [\underline{\underline{C}}][\underline{\underline{B}}]$$

$$\underline{\underline{C}} : \underline{\underline{B}} = a$$

$$\underline{\underline{C}} \otimes \underline{\underline{B}} = 4^{th} \text{ order tensor}$$

$$\sum\sum\sum\sum C_{ij} B_{mn}\, \underline{e}_i \otimes \underline{e}_j \otimes \underline{e}_m \otimes \underline{e}_n$$

$$\left(\sum\sum C_{ij}\, \underline{e}_i \otimes \underline{e}_j \right) : \left(\sum_{mn} B_{mn}\, \underline{e}_m \otimes \underline{e}_n \right)$$

$$\sum_i \sum_j \sum_m \sum_n C_{ij} B_{mn} \delta_{im} \delta_{jn}$$

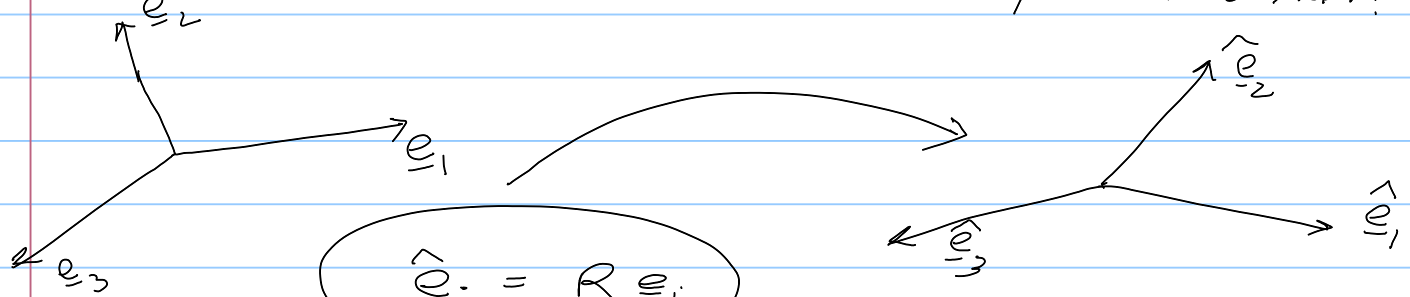
$$= \sum_i \sum_j C_{ij} B_{ij}$$

Rotation tensors:

↓

Axis of rotation $(\underset{\sim}{v})$ and angle of rotation (θ)

Any two sets of orthonormal triads can be relation through a unique rotation tensor!

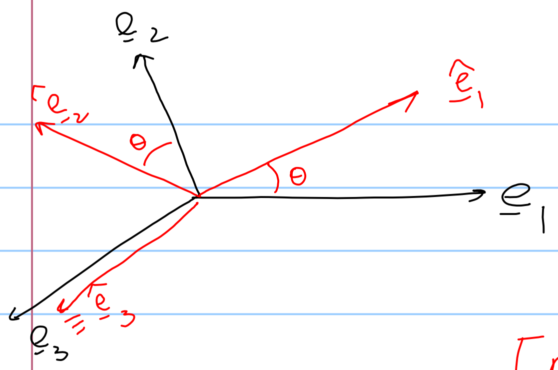


$$\hat{\underline{e}}_i = \underline{\underline{R}}\, \underline{e}_i$$

$$\underline{\underline{R}} = \sum\sum \underset{\sim}{R_{ij}}\, \underline{e}_i \otimes \underline{e}_j$$

$$[\underline{\underline{R}}]_{(\underline{e}_1 - \underline{e}_2 - \underline{e}_3)}$$

$$R_{ij} = (\underline{\underline{R}} \cdot \underline{e}_j) \cdot \underline{e}_i$$



$$= \hat{\underline{e}}_j \cdot \underline{e}_i$$

$$R_{12} = \hat{\underline{e}}_2 \cdot \underline{e}_1 = -\sin\theta$$

$$R_{21} = \hat{\underline{e}}_1 \cdot \underline{e}_2 = \sin\theta$$

$$R_{11} = \hat{\underline{e}}_1 \cdot \underline{e}_1 = \cos\theta$$

$$R_{22} = \hat{\underline{e}}_2 \cdot \underline{e}_2 = \cos\theta$$

$$R_{33} = \hat{\underline{e}}_3 \cdot \underline{e}_3 = 1$$

$$[\underline{\underline{R}}] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$